BV FUNCTIONS ON CARNOT GROUPS
A TENTATIVE BIBLIOGRAPHY

SEBASTIANO NICOLUSSI GOLO

I use the monograph by Ambrosio–Fusco–Pallara both as a reference for the classical theory of BV functions and as a guide for any generalization to non-classical settings:

• L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434

For the theory of rectifiability, I suggest Mattila's book:

• P. Mattila. *Geometry of sets and measures in Euclidean spaces*. Vol. 44. Cambridge Studies in Advanced Mathematics. Fractals and rectifiability. Cambridge University Press, Cambridge, 1995, pp. xii+343

One should also keep an eye on the books by Federer (e.g., for the Lebesgue differentiability theorem) and Hörmander (e.g., for the theory of distributions):

• H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969, pp. xiv+676

• L. Hörmander. *The analysis of linear partial differential operators. I*. vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1983, pp. ix+391

For an introduction to Carnot groups, I suggest the short paper by Le Donne:

• E. Le Donne. "A primer on Carnot groups: homogenous groups, Carnot-Carathéodory spaces, and regularity of their isometries". In: *Anal. Geom. Metr. Spaces* 5 (2017), pp. 116–137

and the monograph by Capogna, Danielli, Pauls and Tyson:

• L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem*. Vol. 259. Progress in Mathematics. Birkhäuser Verlag, Basel, 2007, pp. xvi+223

For a general introduction to BV spaces in Carnot groups, I suggest to read a paper by Franchi:

Date: August 24, 2022.

• B. Franchi. "BV spaces and rectifiability for Carnot-Carathéodory metrics: an introduction". In: *NAFSA 7—Nonlinear analysis, function spaces and applications. Vol. 7.* Czech. Acad. Sci., Prague, 2003, pp. 72–132

Some of the first results about the isoperimetric inequality and the isoperimetric problem in the Heisenberg group:

• P. Pansu. "An isoperimetric inequality on the Heisenberg group". In: Special Issue. Conference on differential geometry on homogeneous spaces (Turin, 1983). 1983, 159–174 (1984)

• P. Pansu. "Une inégalité isopérimétrique sur le groupe de Heisenberg". In: *C. R. Acad. Sci. Paris Sér. I Math.* 295.2 (1982), pp. 127–130

Sobolev spaces, BV functions and various geometric inequalities in CC spaces have been pioneered by the school of Garofalo. I highlight in particular the following works, where you can find proofs of the Isoperimetric and Poincaré inequalities and Sobolev embedding theorems:

• L. Capogna, D. Danielli, and N. Garofalo. "The geometric Sobolev embedding for vector fields and the isoperimetric inequality". In: *Comm. Anal. Geom.* 2.2 (1994), pp. 203–215

• N. Garofalo and D.-M. Nhieu. "Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces". In: *Comm. Pure Appl. Math.* 49.10 (1996), pp. 1081–1144

• D. Danielli, N. Garofalo, and D. M. Nhieu. "Sub-Riemannian calculus on hypersurfaces in Carnot groups". In: *Adv. Math.* 215.1 (2007), pp. 292–378

For the development of the theory of BV functions and of the perimeter, two papers by Ambrosio have been crucial. Ambrosio worked in metric measure spaces and proved, among other results, that the perimeter measure is asymptotically doubling.

• L. Ambrosio. "Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces". In: *Adv. Math.* 159.1 (2001), pp. 51–67

• L. Ambrosio. "Fine properties of sets of finite perimeter in doubling metric measure spaces". In: *Set-Valued Anal.* 10.2-3 (2002). Calculus of variations, nonsmooth analysis and related topics, pp. 111–128

The trio Franchi–Serapioni–Serra Cassano (FSSC) is a protagonist of the theory of perimeter and rectifiability in Carnot groups:

• B. Franchi, R. Serapioni, and F. Serra Cassano. "Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields". In: *Houston J. Math.* 22.4 (1996), pp. 859–890

• B. Franchi, R. Serapioni, and F. Serra Cassano. "On the structure of finite perimeter sets in step 2 Carnot groups". In: *J. Geom. Anal.* 13.3 (2003), pp. 421–466

• B. Franchi, R. Serapioni, and F. Serra Cassano. "Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups". In: *Comm. Anal. Geom.* 11.5 (2003), pp. 909–944

• B. Franchi, R. Serapioni, and F. Serra Cassano. "Intrinsic submanifolds, graphs and currents in Heisenberg groups". In: *Lecture notes of Seminario Interdisciplinare di Matematica. Vol. IV.* Lect. Notes Semin. Interdiscip. Mat., IV. S.I.M. Dep. Mat. Univ. Basilicata, Potenza, 2005, pp. 23–38

• B. Franchi, R. Serapioni, and F. Serra Cassano. "Intrinsic Lipschitz graphs in Heisenberg groups". In: *J. Nonlinear Convex Anal.* 7.3 (2006), pp. 423–441

• B. Franchi and R. P. Serapioni. "Intrinsic Lipschitz graphs within Carnot groups". In: *J. Geom. Anal.* 26.3 (2016), pp. 1946–1994

For the most recent results in the theory of BV functions, I signal:

• V. Magnani. *Elements of geometric measure theory on sub-Riemannian groups.* Scuola Normale Superiore, Pisa, 2002, pp. viii+195

• D. Vittone. *Submanifolds in Carnot groups.* Vol. 7. Tesi. Scuola Normale Superiore di Pisa (Nuova Series) [Theses of Scuola Normale Superiore di Pisa (New Series)]. Thesis, Scuola Normale Superiore, Pisa, 2008. Edizioni della Normale, Pisa, 2008, pp. xx+180

• L. Ambrosio, R. Ghezzi, and V. Magnani. "BV functions and sets of finite perimeter in sub-Riemannian manifolds". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32.3 (2015), pp. 489–517

For the fine properties of BV functions in CC spaces:

• S. Don and D. Vittone. "Fine properties of functions with bounded variation in Carnot-Carathéodory spaces". In: *J. Math. Anal. Appl.* 479.1 (2019), pp. 482–530

• S. Don, A. Massaccesi, and D. Vittone. "Rank-one theorem and subgraphs of BV functions in Carnot groups". In: *J. Funct. Anal.* 276.3 (2019), pp. 687–715

The paper where the technics of iterated blow-ups for sets of finite perimeter has been developed:

• L. Ambrosio, B. Kleiner, and E. Le Donne. "Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane". In: *J. Geom. Anal.* 19.3 (2009), pp. 509–540

An understanding of sets constant normal:

• C. Bellettini and E. Le Donne. "Sets with constant normal in Carnot groups: properties and examples". In: *Comment. Math. Helv.* 96.1 (2021), pp. 149–198

• E. Le Donne and T. Moisala. "Semigenerated Carnot algebras and applications to sub-Riemannian perimeter". In: *Math. Z.* 299.3-4 (2021), pp. 2257–2285

• S. Don, E. Le Donne, T. Moisala, and D. Vittone. "A rectifiability result for finite-perimeter sets in Carnot groups". In: *arXiv e-prints*, arXiv:1912.00493 (Dec. 2019), arXiv:1912.00493. arXiv: `1912.00493 [math.AP]`

References

Finally, I have made a list of the works that I have used more or less. I have likely missed something important: beware! The following list is in chronological order, which could be useful to get a feeling of what the development has been.

[1] J. L. Kelley. "Decomposition and representation theorems in measure theory". In: *Math. Ann.* 163 (1966), pp. 89–94.

[2] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969, pp. xiv+676.

[3] P. Pansu. "Une inégalité isopérimétrique sur le groupe de Heisenberg". In: *C. R. Acad. Sci. Paris Sér. I Math.* 295.2 (1982), pp. 127–130.

[4] L. Hörmander. *The analysis of linear partial differential operators. I.* Vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1983, pp. ix+391.

[5] P. Pansu. "An isoperimetric inequality on the Heisenberg group". In: Special Issue. Conference on differential geometry on homogeneous spaces (Turin, 1983). 1983, 159–174 (1984).

[6] D. Jerison. "The Poincaré inequality for vector fields satisfying Hörmander's condition". In: *Duke Math. J.* 53.2 (1986), pp. 503–523.

[7] L. Capogna, D. Danielli, and N. Garofalo. "The geometric Sobolev embedding for vector fields and the isoperimetric inequality". In: *Comm. Anal. Geom.* 2.2 (1994), pp. 203–215.

[8] P. Mattila. *Geometry of sets and measures in Euclidean spaces.* Vol. 44. Cambridge Studies in Advanced Mathematics. Fractals and rectifiability. Cambridge University Press, Cambridge, 1995, pp. xii+343.

[9] B. Franchi, R. Serapioni, and F. Serra Cassano. "Meyers-Serrin type theorems and relaxation of variational integrals depending on vector fields". In: *Houston J. Math.* 22.4 (1996), pp. 859–890.

[10] N. Garofalo and D.-M. Nhieu. "Isoperimetric and Sobolev inequalities for Carnot-Carathéodory spaces and the existence of minimal surfaces". In: *Comm. Pure Appl. Math.* 49.10 (1996), pp. 1081–1144.

[11] G. B. Folland. *Real analysis.* Second. Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1999, pp. xvi+386.

[12] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.

[13] L. Ambrosio and B. Kirchheim. "Currents in metric spaces". In: *Acta Math.* 185.1 (2000), pp. 1–80.

[14] L. Ambrosio and B. Kirchheim. "Rectifiable sets in metric and Banach spaces". In: *Math. Ann.* 318.3 (2000), pp. 527–555.

[15] E. Lanconelli and D. Morbidelli. "On the Poincaré inequality for vector fields". In: *Ark. Mat.* 38.2 (2000), pp. 327–342.

[16] L. Ambrosio. "Some fine properties of sets of finite perimeter in Ahlfors regular metric measure spaces". In: *Adv. Math.* 159.1 (2001), pp. 51–67.

[17] B. Franchi, R. Serapioni, and F. Serra Cassano. "Rectifiability and perimeter in the Heisenberg group". In: *Math. Ann.* 321.3 (2001), pp. 479–531.

[18] L. Ambrosio. "Fine properties of sets of finite perimeter in doubling metric measure spaces". In: *Set-Valued Anal.* 10.2-3 (2002). Calculus of variations, nonsmooth analysis and related topics, pp. 111–128.

[19] V. Magnani. *Elements of geometric measure theory on sub-Riemannian groups*. Scuola Normale Superiore, Pisa, 2002, pp. viii+195.

[20] L. Ambrosio and V. Magnani. "Weak differentiability of BV functions on stratified groups". In: *Math. Z.* 245.1 (2003), pp. 123–153.

[21] B. Franchi. "BV spaces and rectifiability for Carnot-Carathéodory metrics: an introduction". In: *NAFSA 7—Nonlinear analysis, function spaces and applications. Vol. 7.* Czech. Acad. Sci., Prague, 2003, pp. 72–132.

[22] B. Franchi, R. Serapioni, and F. Serra Cassano. "On the structure of finite perimeter sets in step 2 Carnot groups". In: *J. Geom. Anal.* 13.3 (2003), pp. 421–466.

[23] B. Franchi, R. Serapioni, and F. Serra Cassano. "Regular hypersurfaces, intrinsic perimeter and implicit function theorem in Carnot groups". In: *Comm. Anal. Geom.* 11.5 (2003), pp. 909–944.

[24] G. P. Leonardi and S. Rigot. "Isoperimetric sets on Carnot groups". In: *Houston J. Math.* 29.3 (2003), pp. 609–637.

[25] M. Miranda Jr. "Functions of bounded variation on "good" metric spaces". In: *J. Math. Pures Appl. (9)* 82.8 (2003), pp. 975–1004.

[26] B. Kirchheim and F. Serra Cassano. "Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 3.4 (2004), pp. 871–896.

[27] S. D. Pauls. "Minimal surfaces in the Heisenberg group". In: *Geom. Dedicata* 104 (2004), pp. 201–231.

[28] B. Franchi, R. Serapioni, and F. Serra Cassano. "Intrinsic submanifolds, graphs and currents in Heisenberg groups". In: *Lecture notes of Seminario Interdisciplinare di Matematica. Vol. IV.* Lect. Notes Semin. Interdiscip. Mat., IV. S.I.M. Dep. Mat. Univ. Basilicata, Potenza, 2005, pp. 23–38.

[29] B. Franchi, R. Serapioni, and F. Serra Cassano. "Intrinsic Lipschitz graphs in Heisenberg groups". In: *J. Nonlinear Convex Anal.* 7.3 (2006), pp. 423–441.

[30] L. Capogna, D. Danielli, S. D. Pauls, and J. T. Tyson. *An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem.* Vol. 259. Progress in Mathematics. Birkhäuser Verlag, Basel, 2007, pp. xvi+223.

[31] D. Danielli, N. Garofalo, and D. M. Nhieu. "Sub-Riemannian calculus on hypersurfaces in Carnot groups". In: *Adv. Math.* 215.1 (2007), pp. 292–378.

[32] B. Franchi, R. Serapioni, and F. Serra Cassano. "Regular submanifolds, graphs and area formula in Heisenberg groups". In: *Adv. Math.* 211.1 (2007), pp. 152–203.

[33] V. Magnani and D. Vittone. "An intrinsic measure for submanifolds in stratified groups". In: *J. Reine Angew. Math.* 619 (2008), pp. 203–232.

[34] R. Monti. "Heisenberg isoperimetric problem. The axial case". In: *Adv. Calc. Var.* 1.1 (2008), pp. 93–121.

[35] D. Vittone. *Submanifolds in Carnot groups.* Vol. 7. Tesi. Scuola Normale Superiore di Pisa (Nuova Series) [Theses of Scuola Normale Superiore di Pisa (New Series)]. Thesis, Scuola Normale Superiore, Pisa, 2008. Edizioni della Normale, Pisa, 2008, pp. xx+180.

[36] L. Ambrosio, B. Kleiner, and E. Le Donne. "Rectifiability of sets of finite perimeter in Carnot groups: existence of a tangent hyperplane". In: *J. Geom. Anal.* 19.3 (2009), pp. 509–540.

[37] D. Danielli, N. Garofalo, D. M. Nhieu, and S. D. Pauls. "Instability of graphical strips and a positive answer to the Bernstein problem in the Heisenberg group $\mathbb{H}^1$". In: *J. Differential Geom.* 81.2 (2009), pp. 251–295.

[38] R. Monti and M. Rickly. "Convex isoperimetric sets in the Heisenberg group". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 8.2 (2009), pp. 391–415.

[39] D. Danielli, N. Garofalo, D.-M. Nhieu, and S. D. Pauls. "The Bernstein problem for embedded surfaces in the Heisenberg group $\mathbb{H}^1$". In: *Indiana Univ. Math. J.* 59.2 (2010), pp. 563–594.

[40] A. Hurtado, M. Ritoré, and C. Rosales. "The classification of complete stable area-stationary surfaces in the Heisenberg group $\mathbb{H}^1$". In: *Adv. Math.* 224.2 (2010), pp. 561–600.

[41] P. Mattila, R. Serapioni, and F. Serra Cassano. "Characterizations of intrinsic rectifiability in Heisenberg groups". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 9.4 (2010), pp. 687–723.

[42] B. Franchi, R. Serapioni, and F. Serra Cassano. "Differentiability of intrinsic Lipschitz functions within Heisenberg groups". In: *J. Geom. Anal.* 21.4 (2011), pp. 1044–1084.

[43] V. Magnani. "An area formula in metric spaces". In: *Colloq. Math.* 124.2 (2011), pp. 275–283.

[44] M. Ritoré. "A proof by calibration of an isoperimetric inequality in the Heisenberg group $\mathbb{H}^n$". In: *Calc. Var. Partial Differential Equations* 44.1-2 (2012), pp. 47–60.

[45] C. Bellettini and E. Le Donne. "Regularity of sets with constant horizontal normal in the Engel group". In: *Comm. Anal. Geom.* 21.3 (2013), pp. 469–507.

[46] V. Magnani. "Towards differential calculus in stratified groups". In: *J. Aust. Math. Soc.* 95.1 (2013), pp. 76–128.

[47] L. Ambrosio and S. Di Marino. "Equivalent definitions of BV space and of total variation on metric measure spaces". In: *J. Funct. Anal.* 266.7 (2014), pp. 4150–4188.

[48] B. Franchi, M. Marchi, and R. P. Serapioni. "Differentiability and approximate differentiability for intrinsic Lipschitz functions in Carnot groups and a Rademacher theorem". In: *Anal. Geom. Metr. Spaces* 2.1 (2014), pp. 258–281.

[49] R. Monti. "Isoperimetric problem and minimal surfaces in the Heisenberg group". In: *Geometric measure theory and real analysis.* Vol. 17. CRM Series. Ed. Norm., Pisa, 2014, pp. 57–129.

[50] L. Ambrosio, R. Ghezzi, and V. Magnani. "BV functions and sets of finite perimeter in sub-Riemannian manifolds". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32.3 (2015), pp. 489–517.

[51] M. Galli and M. Ritoré. "Area-stationary and stable surfaces of class C^1 in the sub-Riemannian Heisenberg group $\mathbb{H}^1$". In: *Adv. Math.* 285 (2015), pp. 737–765.

[52] V. Magnani. "On a measure-theoretic area formula". In: *Proc. Roy. Soc. Edinburgh Sect. A* 145.4 (2015), pp. 885–891.

[53] V. Magnani, J. T. Tyson, and D. Vittone. "On transversal submanifolds and their measure". In: *J. Anal. Math.* 125 (2015), pp. 319–351.

[54] L. Ambrosio and R. Ghezzi. "Sobolev and bounded variation functions on metric measure spaces". In: *Geometry, analysis and dynamics on sub-Riemannian manifolds. Vol. II.* EMS Ser. Lect. Math. Eur. Math. Soc., Zürich, 2016, pp. 211–273.

[55] B. Franchi and R. P. Serapioni. "Intrinsic Lipschitz graphs within Carnot groups". In: *J. Geom. Anal.* 26.3 (2016), pp. 1946–1994.

[56] E. Le Donne. “A primer on Carnot groups: homogenous groups, Carnot-Carathéodory spaces, and regularity of their isometries”. In: *Anal. Geom. Metr. Spaces* 5 (2017), pp. 116–137.

[57] S. Golo. “Some remarks on contact variations in the first Heisenberg group”. In: *Ann. Acad. Sci. Fenn. Math.* 43.1 (2018), pp. 311–335.

[58] S. Don, E. Le Donne, T. Moisala, and D. Vittone. “A rectifiability result for finite-perimeter sets in Carnot groups”. In: *arXiv e-prints*, arXiv:1912.00493 (Dec. 2019), arXiv:1912.00493. arXiv: `1912.00493 [math.AP]`.

[59] S. Don, A. Massaccesi, and D. Vittone. “Rank-one theorem and subgraphs of BV functions in Carnot groups”. In: *J. Funct. Anal.* 276.3 (2019), pp. 687–715.

[60] S. Don and D. Vittone. “Fine properties of functions with bounded variation in Carnot-Carathéodory spaces”. In: *J. Math. Anal. Appl.* 479.1 (2019), pp. 482–530.

[61] V. Magnani. “Towards a theory of area in homogeneous groups”. In: *Calc. Var. Partial Differential Equations* 58.3 (2019), Paper No. 91, 39.

[62] S. Nicolussi and F. Serra Cassano. “The Bernstein problem for Lipschitz intrinsic graphs in the Heisenberg group”. In: *Calc. Var. Partial Differential Equations* 58.4 (2019), Art. 141, 28.

[63] F. Corni and V. Magnani. “Area formula for regular submanifolds of low codimension in Heisenberg groups”. In: *arXiv e-prints*, arXiv:2002.01433 (Feb. 2020), arXiv:2002.01433. arXiv: `2002.01433 [math.MG]`.

[64] E. L. Donne and T. Moisala. “Semigenerated step-3 Carnot algebras and applications to sub-Riemannian perimeter”. In: (Apr. 2020). eprint: `2004.08619`.

[65] C. Bellettini and E. Le Donne. “Sets with constant normal in Carnot groups: properties and examples”. In: *Comment. Math. Helv.* 96.1 (2021), pp. 149–198.

[66] S. Eriksson-Bique, C. Gartland, E. Le Donne, L. Naples, and S. Nicolussi-Golo. “Nilpotent groups and biLipschitz embeddings into L^1”. In: *arXiv e-prints*, arXiv:2112.11402 (Dec. 2021), arXiv:2112.11402. arXiv: `2112.11402 [math.MG]`.

[67] S. N. Golo and M. Ritoré. “Area-minimizing cones in the Heisenberg group H”. In: *Ann. Fenn. Math.* 46.2 (2021), pp. 945–956.

[68] E. Le Donne and T. Moisala. “Semigenerated Carnot algebras and applications to sub-Riemannian perimeter”. In: *Math. Z.* 299.3-4 (2021), pp. 2257–2285.

[69] A. Julia, S. Nicolussi Golo, and D. Vittone. “Area of intrinsic graphs and coarea formula in Carnot groups”. In: *Math. Z.* 301.2 (2022), pp. 1369–1406.