Introduction to Partial Differential Equations – Exercises for Week 10 –

1. Sobolev Spaces

Exercise 1.1. For $x \in \mathbb{R}^N$ and $q \in [1, +\infty)$, define

$$\|x\|_q = (\sum_{j=1}^N |x_j|^q)^{1/q}, \text{ and } \|x\|_\infty = \max_{j \le N} |x_j|. \tag{1}$$

Show that for every $q_1, q_2 \in [1, +\infty]$ there is L such that

$$\frac{1}{L}\|x\|_{q_2} \le \|x\|_{q_1} \le L\|x\|_{q_2}. \tag{2}$$

Deduce that all norms on $W^{m,p}(\Omega)$

$$(\sum_{|\alpha| \le m} \|\mathrm{D}^\alpha u\|^q_{L^p(\Omega)})^{1/q}, \text{ for } q \in [1, +\infty), \text{ or } \max_{|\alpha| \le m} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}. \tag{3}$$

are biLipschitz equivalent to $\|u\|_{W^{m,p}(\Omega)}$. ◊

Exercise 1.2. Show that $\|\cdot\|_{W^{m,p}(\Omega)}$ is a norm on $W^{m,p}(\Omega)$. ◊

Exercise 1.3. Show that $W^{m,p}(\Omega)$ is isometric to a closed subspace of $L^p(\Omega)^N$, where $N = \#\{\alpha \in \mathbb{N}^n : |\alpha| \le m\}$. ◊

Exercise 1.4. Inspired by a lemma we proved in class, we can say with no doubt that, if $u_j \to u$ in $W^{m,p}(\Omega)$, then $u_j \to u$ in $\mathscr{D}'(\Omega)$. We may wonder if the converse is also true, that is: is it true that, if $u_j \to u$ in $\mathscr{D}'(\Omega)$ and $u \in W^{m,p}(\Omega)$, then $u_j \to u$ in $W^{m,p}(\Omega)$? If this was a video, you would pause it to think about the question yourself. However, this is not a video: can you stop reading?

The answer is not hard. The point is to find a sequence of functions v_j on $\mathbb{R}$ that converge distributionally to 0, but have L^p norm equal to 1. For instance, take $v_j(x) = \sum_{m=0}^{2^j} \mathbb{1}_{[m2^{-j},(m+1)2^{-j})}(x)$. Then $|v_j| = \mathbb{1}_{[0,1)}$, and $\int_{\mathbb{R}} v_j \phi \,\mathrm{d}x \to 0$ for every $\phi \in \mathscr{D}(\mathbb{R})$ (check it!). We have thus a sequence $v_j \to 0$ in $\mathscr{D}'(\mathbb{R})$ with $\|v_j\|_{L^p} = 1$ for all j. Take $u_j(x) = \int_{-\infty}^x v_j(t)\,\mathrm{d}t$: check that $u_j(2) = 0$. So, u_j is an absolutely continuous function $\mathbb{R} \to \mathbb{R}$ with compact support. Moreover, $u_j \to 0$ distributionally (check it!) but u_j does not converge to 0 in $W^{1,1}(\mathbb{R})$. So, the answer is no. ◊

Exercise 1.5. An example of weird Sobolev functions. For $\beta \in \mathbb{R}$, define $u_\beta : \mathbb{R}^n \to \mathbb{R}$,

$$u_\beta(x) = |x|^\beta. \tag{4}$$

(1) Show that $u_\beta \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ whenever $\beta > -n$.

(2) Show that $u_\beta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $\beta > 1 - n$.

(3) Take an enumeration $\{q_k\}_{k\in\mathbb{N}} = \mathbb{Q}^n \cap B(0,1)$ of points with rational coordinates inside the unit ball. Define

$$w_\beta(x) = \sum_{k=1}^\infty \frac{u_\beta(x - q_k)}{2^k}. \tag{5}$$

Notice that, since $u_\beta \ge 0$, then $w_\beta(x) \in [0, +\infty]$ is well defined for every $x \in \mathbb{R}^n$. Show that, if $\beta > 1 - n$, then $w_\beta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

I want to remark that, if $n \ge 2$, then $1 - n < 0$ and thus we can take $\beta < 0$: in these cases, w_β has a "pole" at every point in $\mathbb{Q}^n \cap B(0,1)$. ◊

Exercise 1.6. Let $j \in \mathbb{Z}$. Show that, if $A \subset B(0, 2^j)$ is such that, for every $a_1, a_2 \in A$ distinct, $B(a_1, 2^{j-4}) \cap B(a_2, 2^{j-4}) = \emptyset$, then $\#A \le 2^{5n}$.

Hint: The union of the pairwise disjoint balls $\{B(a, 2^{j-4})\}_{a\in A}$ is a subset of $B(0, 2^j)$, so its volume... ◊

Exercise 1.7. Show that, for every $p \in [1, \infty)$, the constant function $u \equiv 1$ belongs to $W^{1,p}((0,1))$, but it is not the limit in $W^{1,p}$ of functions $C^\infty_c((0,1))$. ◊

Exercise 1.8. Show that $C^\infty_c(\Omega)$ is dense in $L^p(\Omega)$. ◊

Exercise 1.9. When we prove the *Gagliardo–Nirenberg–Sobolev inequality*, we required $n \ge 2$. What happens when $n = 1$? ◊

Exercise 1.10. Show that, if $u \in C^1_c(\mathbb{R}^n)$ and $\gamma > 1$, then $|u|^\gamma \in C^1_c(\mathbb{R}^n)$ and $\nabla(|u|^\gamma) = \gamma|u|^{\gamma-1}\nabla u$.

Hint: on the set $\{u \ne 0\}$, there the statement is trivial. What remains to show is that, if $u(x) = 0$, then $|u|^\gamma$ is differentiable at x with derivative equal to 0. ◊

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