Introduction to Partial Differential Equations – Exercises for Week 9 –

1. Shwartz class of functions

For $f:\mathbb{R}^n \to \mathbb{C}$ smooth, define

$$p_{\alpha,N}(f) = \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\mathrm{D}^\alpha f(x)|. \tag{1}$$

The space $\mathscr{S}$ is the space of $f\in C^\infty(\mathbb{R}^n)$ with $p_{\alpha,N}(f)<\infty$ for all α, N.

Exercise 1.1. Show that $\mathscr{S}\subset L^p$ for all $p\in[1,+\infty]$.

Hint: Use (1) for N large enough. ◊

Exercise 1.2. Show that $f_j\to f_\infty$ in $\mathscr{S}$ if and only if $x^\alpha \mathrm{D}^\beta f_j \to x^\alpha \mathrm{D}^\beta f_\infty$ in L^∞, for every $\alpha,\beta\in\mathbb{N}^n$. ◊

Exercise 1.3. Show that, if $f_j\to f_\infty$ in $\mathscr{S}$, then, for every $\alpha\in\mathbb{N}^n$, $\mathrm{D}^\alpha f_j\to \mathrm{D}^\alpha f_\infty$ in $L^p(\mathbb{R}^n)$, for all $p\in[1,+\infty]$.

Hint: Take first $p=1$ and $\alpha=0$. Up to substituting f_j with f_j-f_∞, we can also assume $f_\infty=0$. So we need to show that, if $f_j\to 0$ in $\mathscr{S}$, then $f_j\to 0$ in $L^1(\mathbb{R}^n)$. The convergence $f_j\to 0$ in $\mathscr{S}$ implies that, for every $N>0$ and every $\epsilon>0$, there is $J\in\mathbb{N}$ such that $|f_j(x)|(1+|x|)^N<\epsilon$ for all $x\in\mathbb{R}^n$ and all $j>J$. ◊

Exercise 1.4. Define $g_\epsilon:\mathbb{R}^n\to[0,+\infty)$ by

$$g_\epsilon(z) = \frac{1}{\epsilon^n} e^{-\pi|z|^2/\epsilon^2} = \frac{1}{\epsilon^n}\exp(-\pi|z|^2/\epsilon^2). \tag{2}$$

Show that, if $f\in\mathscr{S}$, then $f*g_\epsilon\to f$ in $\mathscr{S}$ as $\epsilon\to 0$.

Hint: Go back to the proof of the first statement in Proposition **??**. The now g_ϵ does not have compact support, but $\int_{\mathbb{R}^n\setminus B(0,1)} g_\epsilon(x)\,\mathrm{d}x$ is arbitrarily small as $\epsilon\to 0$. ◊

Exercise 1.5. Show that $\mathscr{S}$ is dense in $L^1(\mathbb{R}^n)$.

Hint: Given $f\in L^1(\mathbb{R}^n)$, consider $f_j(x)=\psi(x/j)f*\rho_{1/j}(x)$, where $\{\rho_\epsilon\}_{\epsilon>0}$ is a family of mollifiers, and $\psi\in C^\infty_c(\mathbb{R}^n)$ is a function valued in $[0,1]$ with $B(0,1)\subset\{\psi=1\}$. You then need to show that $f_j\in\mathscr{S}$ and that $f_j\to f$ in $L^1(\mathbb{R}^n)$. Use the fact that, for every $\epsilon>0$ there exists $R>0$ such that $\int_{\mathbb{R}^n\setminus B(0,R)}|f(x)|\,\mathrm{d}x<\epsilon$ (this is a direct consequence of integrability). ◊

Exercise 1.6. (Maybe this already appeared before). Show that $\mathscr{D}$ is dense in $L^p(\mathbb{R}^n)$ for all $p\in[1,\infty)$. Deduce that $\mathscr{S}$ is dense in $L^p(\mathbb{R}^n)$ for all $p\in[1,\infty)$.

But then, show also that, if $f_j\to f_\infty$ in $\mathscr{D}$ or $\mathscr{S}$, then $f_j\to f_\infty$ in $L^p(\mathbb{R}^n)$ for all $p\in[1,\infty)$. ◊

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2. Fourier transform

Recall that, for $u\in L^1(\mathbb{R}^n)$ and $\xi\in\mathbb{R}^n$,

$$\hat u(\xi) = \mathscr{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} u(x)\,\mathrm{d}x. \tag{3}$$

Exercise 2.1. Prove the following properties:

(1) If $u\in L^1(\mathbb{R}^n)$, $a\in\mathbb{R}^n$, and $u_a(x)=u(x+a)$ then $\mathscr{F}(u_a)(\xi)=e^{2\pi i a\cdot\xi}\mathscr{F}(u)(\xi)$.

(2) If $u\in L^1(\mathbb{R}^n)$, $T:\mathbb{R}^n\to\mathbb{R}^n$ is linear and invertible, then

$$\mathscr{F}(u\circ T)(\xi) = |\det T|^{-1}\mathscr{F}(u)((T^{-1})^*\xi). \tag{4}$$

(3) If T is a rotation of $\mathbb{R}^n$, then $\mathscr{F}(u\circ T)=\mathscr{F}(u)\circ T$.

Hint: See [2, Proposition (0.21)] ◊

Exercise 2.2. Compute $\mathscr{F}(\bar u)$ in terms of $\mathscr{F}(u)$. (Here $\bar{\cdot}$ denotes the complex conjugate, that is, for $x,y\in\mathbb{R}$, $\overline{x+iy}=x-iy$.) ◊

Exercise 2.3. Compute $\mathscr{F}(u(-x))$ in terms of $\mathscr{F}(u)$. ◊

Exercise 2.4. Fix $f\in\mathscr{S}$ and define $g(x)=\overline{f(-x)}$. Show that $\hat g(\xi)=\overline{\hat f(\xi)}$. ◊

Exercise 2.5. Compute the Fourier transform of the function

$$u(x) = Ae^{-a|x|^2}, \tag{5}$$

for every $A\in\mathbb{C}$ and $a>0$.

Hint:

$$\mathscr{F}(Ae^{-a|x|^2}) = A\left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{\pi^2}{a}|x|^2}. \tag{6}$$

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Recall from Exercise **??**, that the convolution of two functions $f,g\in L^1(\mathbb{R}^n)$ is a well defined function $f*g\in L^1(\mathbb{R}^n)$.

Exercise 2.6. Show that, for every $f,g\in L^1(\mathbb{R}^n)$,

$$\mathscr{F}(f*g) = \mathscr{F}(f)\cdot\mathscr{F}(g). \tag{7}$$

Or, otherwise stated, $(f*g)^\wedge = \hat f\hat g$. ◊

Exercise 2.7. Show that, for every $f,g\in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\hat g(x)\,\mathrm{d}x = \int_{\mathbb{R}^n}\hat f(\xi)g(\xi)\,\mathrm{d}\xi. \tag{8}$$

Hint: Notice that, by (**??**), $f\hat g\in L^1(\mathbb{R}^n)$ and $\hat f, g\in L^1(\mathbb{R}^n)$. So, unpack the definition of $\hat g$ and use Fubini. ◊

Exercise 2.8. Define $\mathcal{I}:\mathscr{S}\to\mathscr{S}$, $\mathcal{I}(f)(x)=f(-x)$. Show that

$$\mathscr{F}^2=\mathcal{I} \quad\text{and}\quad \mathscr{F}^4=\mathrm{Id}_{\mathscr{S}}. \tag{9}$$

Hint: Forget about the Fourier transform, but only use $\mathscr{F}\mathcal{I}\mathscr{F}=1$. ◊

3. Tempered distributions and their Fourier transform

Exercise 3.1. Show that

$$\mathscr{D}' \supset \mathscr{S}' \supset \mathscr{E}'. \tag{10}$$

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Exercise 3.2. Show the following properties of tempered distributions:

(1) If $u \in \mathscr{S}'$ and $\alpha \in \mathbb{N}^n$, then $\mathrm{D}^\alpha u \in \mathscr{S}'$.
(2) If $u \in \mathscr{S}'$ and $f \in C^\infty(\mathbb{R}^n)$ is such that, for all $\alpha \in \mathbb{N}^n$, $\mathrm{D}^\alpha f$ grows at most polynomially at infinity, then $fu \in \mathscr{S}'$.
(3) If $u \in \mathscr{S}'$ and $f \in \mathscr{S}$, then $u * f \in \mathscr{S}$.

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Exercise 3.3. Show that, if $h : \mathbb{R}^n \to \mathbb{C}$ is a measurable function that grows at most polynomially, then $f \mapsto \int_{\mathbb{R}^n} h(x)f(x)\,\mathrm{d}x$ defines a tempered distribution. ◊

We define the Fourier transform of a tempered distribution $u \in \mathscr{S}'$ as $\hat{u} = \mathscr{F}(u)$, where

$$\hat{u}[f] = u[\hat{f}], \qquad \forall f \in \mathscr{S}. \tag{11}$$

Exercise 3.4. Show the following properties of the Fourier transform of tempered distributions:

(1) If $u \in \mathscr{S}'$, then $\hat{u} \in \mathscr{S}'$.
(2) If $u \in \mathscr{S}'$ is actually in $\mathscr{S}$, i.e., $u[f] = \int_{\mathbb{R}^n} u(x)f(x)\,\mathrm{d}x$ for all $f \in \mathscr{S}$, then $\hat{u}$ as distribution is equal to $\hat{u}$ as function.
(3) $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is a continuous, invertible linear operator, with inverse $\mathscr{F}^{-1}u[f] = u[\mathscr{F}^{-1}(f)]$.
(4) If $u \in \mathscr{S}'$ and $f \in \mathscr{S}$, then

$$\mathscr{F}(u * f) = \hat{u}\hat{f}. \tag{12}$$

(5) If $f \in \mathscr{S}'$, then $\hat{f} \in \mathscr{S}'$ and, for every $\alpha \in \mathbb{N}^n$,

$$\mathscr{F}(\mathrm{D}^\alpha f) = (2\pi i \xi)^\alpha \mathscr{F}(f), \tag{13}$$

$$\mathscr{F}((-2\pi i x)^\alpha f) = \mathrm{D}^\alpha \mathscr{F}(f). \tag{14}$$

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Exercise 3.5. Compute $\mathscr{F}(\delta_0)$. ◊

Exercise 3.6. Compute $\mathscr{F}(1)$. ◊

Exercise 3.7. Compute $\mathscr{F}(p(x))$, where $p(x) = \sum_{|\alpha| \le N} c_\alpha x^\alpha$ is a polynomial of degree N. ◊

Remark 3.1. It is a fact that, if $\hat{u}$ has compact support, then u is analytic, see [3, Thm 7.1.14] It is a recurrent theme that regularity of u is proportional to integrability of $\hat{u}$ (and viceversa, of course). There are two sorts of "equilibrium points" of this behavior: $\mathscr{S}$ and $L^2(\mathbb{R}^n)$.

4. Applications to PDE

Exercise 4.1. Assuming $g, h \in \mathscr{S}$, write $u(t)$ from the formula

$$\hat{u}(t) = \cos(2\pi i|\xi|t)\hat{g} + \frac{\sin(2\pi i|\xi|t)}{2\pi i|\xi|}\hat{h}. \tag{15}$$

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Exercise 4.2 (Bessel Potentials)**.** Using the Fourier transform, give a representation to the solutions of

$$-\triangle u + u = f. \tag{16}$$

Hint: See [1, §4.3, p.191]. ◊

Exercise 4.3 (Eigenfunctions of the Laplacian)**.** Using the Fourier transform, give a representation to the solutions of

$$-\triangle u = \lambda u, \tag{17}$$

for $\lambda \in \mathbb{C}$. For which λ there exists a solution? ◊

References

[1] L. C. Evans. *Partial differential equations*. Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749.
[2] G. B. Folland. *Introduction to partial differential equations*. Second. Princeton University Press, Princeton, NJ, 1995, pp. xii+324.
[3] L. Hörmander. *The analysis of linear partial differential operators. I*. Vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1983, pp. ix+391.