Introduction to Partial Differential Equations – Exercises for Week 8 –

1. Distribution with compact support

Exercise 1.1. Check whether the topology on $\mathscr{E}(\Omega)$ is the the initial topology induced by the maps $C^\infty(\Omega) \hookrightarrow C^m(\Omega)$.

Hint: I actually don't know if this is true. So, please send me an email with the result, if you don't mind. ◊

Exercise 1.2. Show that $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$. ◊

Exercise 1.3. Find a sequence $\phi_j \in \mathscr{D}(\Omega)$ that converges to some f in $\mathscr{E}(\Omega)$ but it does not converge in $\mathscr{D}(\Omega)$.

Just as a note: there is a notion of "Cauchy sequence" for topological vector spaces (see Rudin's book). With such a notions available, one could check that $\mathscr{D}(\Omega)$ is complete in its own topology, but that its completion in the topology of $\mathscr{E}(\Omega)$ is $\mathscr{E}(\Omega)$. This should clarify the situation. ◊

Exercise 1.4. In the proof of

$$\mathscr{E}'(\Omega) = \{A \in \mathscr{D}'(\Omega) \text{ with compact support}\}. \tag{1}$$

we have defined the extension of $A \in \mathscr{D}'(\Omega)$ to $\mathscr{E}(\Omega)$ as $A_\psi[f] = A[\psi f]$, where $\psi \in C^\infty_c(\Omega)$ such that $\mathrm{spt} A \subset \mathrm{interior}\{\psi = 1\}$. It looks like this extension depends on the choice of ψ. Does it? ◊

Exercise 1.5. Show that all distributions in $\mathscr{E}'(\Omega)$ have finite order. More precisely, if $u \in \mathscr{E}'(\Omega)$, then there are $N \in \mathbb{N}$ and $C \in \mathbb{R}$ such that, for every $f \in \mathscr{E}(\Omega)$,

$$|u[f]| \le C \|f\|_{C^N(\mathrm{spt}(u))}. \tag{2}$$

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2. Convolution

Exercise 2.1. Prove the following proposition: If $u \in \mathscr{E}'$ and $f \in \mathscr{E}$, then $u * f \in \mathscr{E}$. Moreover

$$\mathrm{spt}(u * f) \subset \mathrm{spt}(u) + \mathrm{spt}(f), \text{ and} \tag{3}$$

$$\mathrm{D}^\alpha(u * f) = u * (\mathrm{D}^\alpha f) = (\mathrm{D}^\alpha u) * f, \qquad \forall \alpha \in \mathbb{N}^n. \tag{4}$$

In particular, if $f \in \mathscr{D}$, then $u * f \in \mathscr{D}$. ◊

Exercise 2.2. Prove the following three statements:

(1) If $\phi_j \to \phi_\infty$ in $\mathscr{D}$ and $u \in \mathscr{D}'$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{E}$.
(2) If $\phi_j \to \phi_\infty$ in $\mathscr{E}$ and $u \in \mathscr{E}'$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{E}$.
(3) If $\phi_j \to \phi_\infty$ in $\mathscr{D}$ and $u \in \mathscr{E}'$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{D}$.

Hint: I have written two out of three of these proofs in my notes. ◊

Exercise 2.3. Show the following properties of convolution of distributions:

(1) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, then $u_1 * u_2 \in \mathscr{D}'$ is well defined.
(2) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, then $u_1 * u_2 = u_2 * u_1$.
(3) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, then $\mathrm{spt}(u_1 * u_2) \subset \mathrm{spt}(u_1) + \mathrm{spt}(u_2)$.
(4) If If $u_1, u_2, u_3 \in \mathscr{D}'$, two of which have compact support, then $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$.
(5) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, and $\alpha \in \mathbb{N}^n$, then $\mathrm{D}^\alpha(u * v) = (\mathrm{D}^\alpha u) * v = u * (\mathrm{D}^\alpha v)$.
(6) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, and one of which is smooth, then $u_1 * u_2 \in \mathscr{E}$.

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Exercise 2.4. Show that $\delta_0 * u = u$ for every $u \in \mathscr{D}'$. What is $\delta_v * u$? ◊

3. Singular support

Exercise 3.1. Show that, if $u \in \mathscr{D}'(\Omega)$, the restriction of u to $\Omega \setminus \mathrm{singSpt}(u)$ is a smooth function. ◊

Exercise 3.2. Show that $\mathrm{singSpt}(u) \subset \mathrm{spt}(u)$. ◊

Exercise 3.3. Show that, if $E \in \mathscr{D}'$ is such that $\mathrm{singSpt}(E) \subset \{0\}$, then $\mathrm{singSpt}(E * u) \subset \mathrm{singSpt}(u)$ for all u that can be convoluted with E. ◊

4. Fundamental solutions

Exercise 4.1. Show that the function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & \text{if } n = 2, \\ \frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{if } n \ge 3, \end{cases} \tag{5}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$, is a fundamental solution of $P = -\triangle$. ◊

Exercise 4.2. Show that the function $\Phi : \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to [0, +\infty)$ defined by (6)

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ 0 & \text{otherwise, i.e., } (x,t) \in (\mathbb{R}^n \times (-\infty, 0]) \setminus \{(0,0)\}, \end{cases}$$

is a fundamental solution of $P = \partial_t - \triangle$. ◊

Exercise 4.3. Find a fundamental solution for the wave operator $P = \Box = \partial_t^2 - \triangle$ in dimension 1, 2 and 3. ◊

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