Introduction to Partial Differential Equations – Exercises for Week 6 –

Our main reference is Chapter 6 of Rudin's book:

• W. Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424

There, you can find even more interesting exercises.

1. Homogeneous wave equation

Exercise 1.1. Recover Kirchhoff's formula from

$$u(x,t) = \frac{1}{\gamma_n}\Bigg[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} g(y)\,\mathrm{d}S(y)\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\fint_{\partial B(x,t)} h(y)\,\mathrm{d}S(y)\right)\Bigg], \tag{1}$$

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Exercise 1.2. Recover Poisson's formula from

$$u(x,t) = \frac{1}{\beta_n}\Bigg[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n}\fint_{B(x,t)} \frac{g(y)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}S(y)\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n}\fint_{B(x,t)} \frac{h(y)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}S(y)\right)\Bigg], \tag{2}$$

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2. Nonhomogeneous wave equation

Exercise 2.1. Prove the following Theorem 2.1.

Theorem 2.1 (Nonhomogeneous equation with null initial data)**.** *Let $n \geq 2$ and $f \in C^{\lfloor \frac{n}{2}\rfloor+1}(\mathbb{R}^n \times [0,+\infty))$. For every $s > 0$, let $u_s : \mathbb{R}^n \times [s,+\infty) \to \mathbb{C}$ be the solution in $C^2(\mathbb{R}^n \times [0,+\infty))$ to*

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^n \times (s,+\infty), \\ u = 0,\ \partial_t u = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{s\}. \end{cases} \tag{3}$$

Define $u : \mathbb{R}^n \times [0,+\infty) \to \mathbb{C}$ by

$$u(x,t) = \int_0^t u_s(x,t)\,\mathrm{d}s. \tag{4}$$

Then $u \in C^2(\mathbb{R}^n \times [0,+\infty))$ and u is a solution to

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = f & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u = 0,\ \partial_t u = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{5}$$

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Exercise 2.2. Write explicitly u from Theorem 2.1 for $n = 2$ and $n = 3$. ◊

Exercise 2.3. Prove the following Theorem 2.2.

Theorem 2.2 (Nonhomogeneous wave equation)**.** *Let $n \geq 2$ and $m = \lfloor \frac{n}{2}\rfloor + 1$. Let $f \in C^m(\mathbb{R}^n \times [0,+\infty))$, Let $g \in C^{m+1}(\mathbb{R}^n)$, and $h \in C^m(\mathbb{R}^n)$.*

*Let u_0 be the function given by Theorem **??** and **??**, and u_1 the function given by Theorem 2.1. Set $u = u_0 + u_1$. Then $u \in C^2(\mathbb{R}^n \times [0,+\infty))$ and u is a solution to*

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = f & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{6}$$

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3. Test functions

Exercise 3.1. We can see $\mathscr{D}(\Omega)$ as a subspace of $\mathscr{D}(\mathbb{R}^n)$, but not as a closed subspace. Why? ◊

Exercise 3.2. Show that the following three topologies on $\mathscr{D}(\Omega)$ are the same:

(1) The first way to construct the topology of $\mathscr{D}(\Omega)$ is defining the collection β of all convex balanced sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}(K) \cap W$ is open $\mathscr{D}(K)$ for all $K \subset \Omega$ compact. A set W is *balanced* if $\lambda W \subset W$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$. The collection β induces a topology τ make of unions of elements of $\{x + W : x \in \mathscr{D}(K),\ w \in \beta\}$. Then τ makes $\mathscr{D}(\Omega)$ into a locally convex topological vector space.

(2) We can endow $C_c^\infty(K) = \bigcap_{m\in\mathbb{N}} C_c^m(K)$ with the initial topology induced by the functions $C_c^\infty(K) \hookrightarrow C_c^m(K)$, and then $\mathscr{D}(\Omega) = \bigcup_{K\Subset\Omega} C_c^\infty(K)$ with the final topology induced by the functions $C_c^\infty(K) \hookrightarrow C_c^\infty(\Omega)$.

(3) We can endow $C_c^m(\Omega) = \bigcup_{K\Subset\Omega} C_c^m(K)$ with the final topology induced by the functions $C_c^m(K) \hookrightarrow C_c^m(\Omega)$, and then $\mathscr{D}(\Omega) = \bigcap_{m\in\mathbb{N}} C_c^m(\Omega)$ with the initial topology induced by the functions $C_c^\infty(\Omega) \hookrightarrow C_c^m(\Omega)$.

If needed, here are the definitions of initial and final topology:

Definition 3.1 (Initial, or projective, topology)**.** Given a set Y and a family of topological spaces $\{Z_i\}_{i\in I}$ and functions $f_i : Y \to Z_i$. The *initial topology* or *projective topology* induced by the family of functions f_i is the coarsest (i.e., smallest) topology in Y that makes all functions f_i continuous.

March 26, 2025. Last git commit: 08d061a in branch: master

Definition 3.2 (Final, or inductive, topology)**.** Given a set Y and a family of topological spaces $\{X_i\}_{i\in I}$ and functions $f_i : X_i \to Y$. The *final topology* or *inductive topology* induced by the family of functions f_i is the finest (i.e., largest) topology in Y that makes all functions f_i continuous.

$\Diamond$

Exercise 3.3. Prove the following proposition:

Proposition 3.3. *Let Y be a locally convex space and $L : \mathscr{D}(\Omega) \to Y$ linear. Then the following are equivalent:*

(1) L is continuous;
(2) if $\phi_j \to 0$ in $\mathscr{D}(\Omega)$ then $L\phi_j \to 0$ in Y;
(3) the restrictions of L to every $C_c^\infty(K) \subset \mathscr{D}(\Omega)$, for $K \Subset \Omega$, are continuous.

$\Diamond$

4. Distributions

Exercise 4.1. Find a sequence $f_j \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $\|f_j\|_{L^1([0,1])} = 1$ but $A_{f_j} \to 0$ in $\mathscr{D}'(\mathbb{R})$. $\Diamond$

Exercise 4.2. Show that $f_j \to 0$ weakly* in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, if and only if $A_{f_j} \to 0$ in $\mathscr{D}'(\mathbb{R}^n)$. The weak* convergence is $\int_{\mathbb{R}} f_j g \,\mathrm{d}x \to 0$ for all $g \in L^\infty(\mathbb{R}^n)$ with compact support. $\Diamond$

Exercise 4.3. Let $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega)$ be a sequence of harmonic functions and suppose that $u_k \to A$ in $\mathscr{D}'(\Omega)$. Show that A is a harmonic function. $\Diamond$

Exercise 4.4. Show that $\mathscr{D}(\Omega)$ is dense in $(C_0(\Omega), \|\cdot\|_{L^\infty})$, where

$$C_0(\Omega) = \{f \in C(\Omega) : \text{for every } \epsilon > 0 \text{ the set } \{|f| \geq \epsilon\} \text{ is compact}\}. \tag{7}$$

$\Diamond$

References

[1] W. Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424.