Introduction to Partial Differential Equations – Exercises for Week 5 –

1. Examples

Exercise 1.1. Compute $\Box u$ for $u(x,t) = \exp((a+bi)\cdot x + \phi t)$, where $a, b \in \mathbb{R}^n$ and $\phi \in \mathbb{R}$.

Exercise 1.2. Show that $\Box$ is invariant under the Lorentz group of transformations. Recall that the Lorentz group is the group of linear automorphisms of the Minkowski space. More explicitely, if M is the $(n+1)\times(n+1)$ matrix

$$M = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

then the Minkowski space is $(\mathbb{R}^n \times \mathbb{R}, M)$ and the Lorentz group is made of matrices $A \in \mathtt{GL}(\mathbb{R}^n \times \mathbb{R})$ such that $A^T M A = M$.

Hint. If it seems too hard, try to solve the exercise at leas for $n = 1$.

Exercise 1.3. For which $a, b \in \mathbb{C}$ we have $\Box u_{a,b} = 0$?

Hint: $a = b$ or $a = -b$.

Exercise 1.4. For which $a, b \in \mathbb{C}$ we have $u_{a,b}$ 1-periodic (i.e., $u_{a,b}(0) = u_{a,b}(1)$) and $\Box u_{a,b} = 0$?

Hint: $a, b \in 2\pi i\mathbb{Z}$ with $a = b$ or $a = -b$.

Exercise 1.5. For every $k \in \mathbb{N}$ and $\ell > 0$, find $u : [0,\ell]\times\mathbb{R} \to \mathbb{R}$ such that

(1) $\Box u = 0$,
(2) $u(0,t) = u(\ell,t) = 0$ for all t, and
(3) there are $0 \le x_0 < x_1 < \cdots < x_k \le \ell$ such that $u(x_j,t) = 0$ for all j and all t.

These functions are the Harmonics of the string pinched at the two ends.

Exercise 1.6. Let $c \in \mathbb{C}$. Find $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ such that

$$\Box u = cu. \tag{2}$$

2. Uniqueness

Exercise 2.1. Let $\hat{x} \in \mathbb{R}^n$ and $R > 0$ and define

$$K(\hat{x}, R) = \{(x,t) \in \mathbb{R}^n \times [0,R] : |x| \le R\}. \tag{3}$$

Prove the following uniqueness result: if $u_1, u_2 \in C^2(K(\hat{x},R))$ are such that $\Box u_1 = \Box u_2$ in $K(\hat{x},R)$ and $u_1 = u_2$ on $K(\hat{x},R) \cap \mathbb{R}^n \times \{0\}$, then $u_1 = u_2$.

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Exercise 2.2. Suppose $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ is a solution to

$$\begin{cases} \Box u = 0 & \text{in } \mathbb{R}^n \times \mathbb{R}; \\ u = g, \ u = h & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{4}$$

Suppose that the support of g is contained in $B(0,r)$, and suppose you are sitting in x. When do you see something happening to u?

3. Wave equation in dimension one

Exercise 3.1. Prove Theorem 3.1:

Theorem 3.1 (Solution to $\Box = 0$ for $n = 1$)**.** *Let $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Define $u : \mathbb{R}\times\mathbb{R} \to \mathbb{C}$ by D'Alambert's formula*

$$u(x,t) = \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(r)\,\mathrm{d}r. \tag{5}$$

Then

(1) $u \in C^2(\mathbb{R}\times\mathbb{R})$;
(2) $\partial_t^2 u - \partial_x^2 u = \Box u = 0$ in $\mathbb{R}\times\mathbb{R}$;
(3) for every $\hat{x} \in \mathbb{R}$, $u(\hat{x},0) = g(\hat{x})$ and $\partial_t u(\hat{x},0) = h(\hat{x})$.

Moreover, u is the only solution to

$$\begin{cases} \Box u = 0 & \text{in } \mathbb{R} \times (0,+\infty), \\ u = g, \ \partial_t u = h & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{6}$$

Exercise 3.2. Try to solve the nonhomogeneous version of (6), that is,

$$\begin{cases} \Box u = f & \text{in } \mathbb{R} \times (0,+\infty), \\ u = g, \ \partial_t u = h & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{7}$$

for some given f.

Exercise 3.3. Solutions to the wave equation given by D'Alambert's formula (5) are of the form

$$u(x,t) = F(x+t) + G(x-t), \tag{8}$$

with $F, G \in C^2(\mathbb{R})$. The two functions represent a forward-moving wave and a backward-moving wave. Can you say which is which?

Exercise 3.4. Prove Theorem 3.2:

Theorem 3.2. *Let $\mathbb{R}_+ = (0,+\infty)$. Let $g \in C^2(\bar{\mathbb{R}}_+)$ and $h \in C^1(\bar{\mathbb{R}}_+)$ be such that $g(0) = h(0) = g''(0) = 0$. The function $u : \mathbb{R}_+ \times [0,+\infty) \to \mathbb{C}$,*

$$u(x,t) = \begin{cases} \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y & \text{if } 0 \le t \le x, \\ \frac{1}{2}(g(x+t) - g(x-t)) + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,\mathrm{d}y & \text{if } 0 \le x \le t, \end{cases} \tag{9}$$

belongs to $C^2(\mathbb{R}_+ \times (0,+\infty)) \cap C^0(\bar{\mathbb{R}}_+ \times [0,+\infty))$ *and it is THE solution to*

$$\begin{cases} \Box u = 0 & \text{in } \mathbb{R}_+ \times (0,+\infty), \\ u(x,0) = g(x),\ \partial_t u(x,0) = h(x) & \forall x \in \mathbb{R}_+, \\ u(0,t) = 0 & \forall t > 0. \end{cases} \tag{10}$$

Exercise 3.5. For $F, G \in C^2(\mathbb{R})$, define $\tilde{u}(x,t) = F(x+t) + G(x-t)$. Then we know that $\Box\tilde{u}$, see Exercise 3.3. For which F and G we have $\tilde{u}(0,t) = 0$ for all t? Solve (10) finding the correct F and G.

Exercise 3.6. Solve the string problem

$$\begin{cases} \Box u = 0 & \text{in } [0,1] \times (0,+\infty), \\ u(x,0) = g(x),\ \partial_t u(x,0) = h(x) & \forall x \in [0,1], \\ u(0,t) = u(1,t) = 0 & \forall t > 0. \end{cases} \tag{11}$$

You need to find some conditions on g and h to make u of class C^2: it is part of the exercise.

Next, for every $n \in \mathbb{N}$, find $u_n \in C^2([0,1] \times \mathbb{R})$ such that $\Box u_n = 0$ and $u_n(k/n, t) = 0$ for all $k \in \{1, \dots, n\}$. These functions u_n are called *harmonics* of the string.

4. Wave equation in dimension three

Exercise 4.1. Prove Theorem 4.1:

Theorem 4.1 (Kirchhoff's formula)**.** *Let* $u \in C^2(\mathbb{R}^3 \times [0,+\infty))$ *be such that* $\Box u = 0$ *in* $\mathbb{R}^3 \times (0,+\infty)$. *Then, for every* $s \geq 0$ *and every* $t > 0$,

$$u(x, s+t) = \fint_{\partial B(x,t)} \left(u(y,s) + \nabla_y u(y,s) \cdot (y-x) + t\partial_s u(y,s)\right) \mathrm{d}S(y). \tag{12}$$

Vice-versa, let $g \in C^2(\mathbb{R}^3)$ *and* $h \in C^1(\mathbb{R}^2)$, *and define*

$$u(x,t) = \fint_{\partial B(x,t)} \left(g(y) + \nabla_y g(y) \cdot (y-x) + th(y)\right) \mathrm{d}S(y). \tag{13}$$

Then $u \in C^2(\mathbb{R}^3 \times [0,+\infty))$ *and* u *is a solution to*

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^3 \times (0,+\infty), \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases} \tag{14}$$