Introduction to Partial Differential Equations – Exercises for Week 3 –

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1. Random exercises

Exercise 1.1. Let $a, b < 0$ and define $\phi : (0,1) \to \mathbb{R}$, $\phi(\theta) = \theta^a \theta^b$. Find the minimum and the point of minimum of ϕ on $(0,1)$.

Exercise 1.2. Show the Multinomial Theorem:

$$\left(\sum_{j=1}^{n} x_j\right)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!} x^\alpha. \tag{1}$$

Exercise 1.3 (Hard: you need to think.)**.** Prove Liouville's Theorem using only the ball mean-value property.

Exercise 1.4. From the proof of the *Representation formula using Green's function* show the following identities:

$$\lim_{\epsilon\to 0} \int_{\partial B(x,\epsilon)} G(x,y)\nabla u(y) \cdot \frac{y-x}{|y-x|}\,\mathrm{d}y = 0,$$

$$\lim_{\epsilon\to 0} \int_{\partial B(x,\epsilon)} u(y)\nabla_y \Phi(y-x) \cdot \frac{y-x}{|y-x|}\,\mathrm{d}y = -u(x),$$

$$\lim_{\epsilon\to 0} \int_{\partial B(x,\epsilon)} u(y)\nabla_y \phi^x(y) \cdot \frac{y-x}{|y-x|}\,\mathrm{d}y = 0.$$

Exercise 1.5. Find the Green function for $U = B(0,1)$, the unit ball in $\mathbb{R}^n$.

Hint: Look in Evans' book. To do less work, take his formula for G and check that it is really the Green function.

Exercise 1.6. Prove the following Fundamental Theorem of Calculus of Variations 1.1.

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Theorem 1.1 (Fundamental Theorem of Calculus of Variations)**.** *If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is such that*

$$\forall \phi \in C_c^\infty(U), \qquad \int_U \phi \cdot u \,\mathrm{d}x = 0, \tag{2}$$

then $u = 0$ almost everywhere in $\mathbb{R}^n$.

2. Heat Operator

Exercise 2.1. Define

$$\psi_+ : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}, \qquad \psi_+(x,t) = \frac{1}{t^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right). \tag{3}$$

Show that $(\partial_t - \triangle)\psi_+ = 0$ in $\mathbb{R}^n \times (0,+\infty)$. Draw a graph of $x \mapsto \psi_+(x,t)$ for positive t when $n = 1$ (We will see that this function represents a forward propagation: this is why we have a plus.)

Exercise 2.2. Define

$$\psi_- : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}, \qquad \psi_-(x,t) = \frac{1}{t^{n/2}} \exp\left(\frac{|x|^2}{4t}\right). \tag{4}$$

Show that $(\partial_t - \triangle)\psi_+ = 0$ in $\mathbb{R}^n \times (0,+\infty)$. Draw a graph of $x \mapsto \psi_+(x,t)$ for positive t when $n = 1$. (We will see that this function represents a backward propagation: this is why we have a minus.)

Exercise 2.3 (Symmetries of the heat operator)**.** Show the identity (5): Let $U \subset \mathbb{R}^n$ open, $I \subset \mathbb{R}$ an open interval, $u \in C^2(U \times I)$, $O \in \mathsf{O}(n)$ and $b \in \mathbb{R}^n$, $\tau \in \mathbb{R}$. Define $\tilde{u}(y,s) = u(Oy + b, s + \tau)$. Then $\tilde{u} \in C^2(O^{-1}(U-b) \times (I - \tau))$ and

$$(\partial_t - \triangle)\tilde{u}(y,t) = (\triangle u)(Oy + b, s + \tau). \tag{5}$$

Exercise 2.4 (More symmetries of the heat operator)**.** Let $U \subset \mathbb{R}^n$ open, $I \subset \mathbb{R}$ an open interval, $u \in C^2(U \times I)$, $O \in \mathsf{O}(n)$ and $b \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $\lambda, \sigma \in \mathbb{R} \setminus \{0\}$ Define

$$\tilde{u}(y,s) = u(\lambda O y + b, \sigma s + \tau).$$

Compute $(\partial_t - \triangle)\tilde{u}$ in terms of $(\partial_t - \triangle)u$. Determine for which choices of transformations we have that, if u is a solution to the heat equation, i.e., $(\partial_t - \triangle)u = 0$, then $\tilde{u}$ is also a solution to the heat equation, i.e., $(\partial_t - \triangle)\tilde{u} = 0$.

Exercise 2.5 (Symmetries of the heat operator...)**.** Show that, if $\lambda > 0$ and if $(\partial_t - \triangle)u = 0$, then $(\partial_t - \triangle)\tilde{u} = 0$, where $\tilde{u}(x,t) = u(\lambda x, \lambda^2 t)$.

3. Fundamental solution for the heat operator

Recall that the *fundamental solution for the heat operator* is the function $\Phi : \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to [0,+\infty)$ defined by

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ 0 & \text{otherwise, i.e., } (x,t) \in (\mathbb{R}^n \times (-\infty, 0]) \setminus \{(0,0)\}. \end{cases} \tag{6}$$

Exercise 3.1. Find the formula for the fundamental solution for the heat equation by yourself in the following way: look for Φ of the form $\Phi(x,t) = \frac{1}{t^\alpha} v\left(\frac{|x|}{t^\beta}\right)$, or $\Phi(x,t) = \frac{1}{t^\alpha} v\left(\frac{|x|^2}{t^\beta}\right)$, with $(\partial_t - \triangle_x)\Phi = 0$ in $\mathbb{R}^n \times (0,+\infty)$.

Exercise 3.2. Find a proof (by yourself or in the literature) for

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}. \tag{7}$$

Exercise 3.3 (Smoothness of the fundamental solution)**.** Complete the proof of smoothness of the fundamental solution. Specifically, define

$$\mathscr{F} = \left\{ \phi : \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to \mathbb{R} : \begin{array}{l} \exists P(x,t) \text{ polynomial with, for } t>0, \\ \phi(x,t) = P(x,t^{-1/2}) \exp\left(-\frac{|x|^2}{4t}\right), \\ \text{while } \phi(x,t) = 0 \text{ for } t \le 0. \end{array} \right\}. \tag{8}$$

Then show

(1) $\mathscr{F} \subset C^0(\mathbb{R}^n \times \mathbb{R})$.
(2) if $\phi \in \mathscr{F}$, then $\frac{\partial \phi}{\partial x_j}, \frac{\partial \phi}{\partial t} \in \mathscr{F}$, for all $j \in \{1,\dots,n\}$.

Conclude that $\mathscr{F} \subset C^\infty(\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\})$.

Exercise 3.4 (Derivatives of the fundamental solution)**.** Show that for every $t > 0$ and $x \in \mathbb{R}^n$,

$$\nabla_x \Phi(x,t) = -\frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) \frac{x}{2t} = -\Phi(x,t)\frac{x}{2t},$$
$$\partial_t \Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) \left(\frac{|x|^2}{4t^2} - \frac{n}{2}\frac{1}{4\pi t}\right) = \Phi(x,t)\left(\frac{|x|^2}{4t^2} - \frac{n}{2t}\right),$$
$$\mathrm{D}^2_x \Phi(x,t) = \Phi(x,t)\left(\frac{x \otimes x}{4t^2} - \frac{\mathrm{Id}}{2t}\right),$$

where Φ is the fundamental solution of the heat operator. Conclude that $(\partial_t - \triangle_x)\Phi = 0$ in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$.

4. Solution to the Cauchy problem

Exercise 4.1 (Smoothness of solution to the Cauchy problem)**.** Fix $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Consider $\mathscr{F}$ as in (8). For $\phi \in \mathscr{F}$, define

$$K_\phi(x,t;y) = \phi(x-y,t)g(y).$$

Show that for every $\phi \in \mathscr{F}$, $\epsilon > 0$ and $R > 0$, there exists a function $h_{R,\epsilon,\phi} \in L^1(\mathbb{R}^n)$ such that

$$|K_\phi(x,t;y)| \le h_{R,\epsilon,\phi}(y) \qquad \forall (x,t,y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \text{ with } |x| \le R \text{ and } t > \epsilon.$$

Conclude that the function

$$u_\phi : x \mapsto \int_{\mathbb{R}^n} \phi(x-y,t)g(y)\,\mathrm{d}y$$

belongs to $C^\infty(\mathbb{R}^n \times (0,+\infty))$. Moreover, conclude that $(\partial_t - \triangle)u = 0$ in $\mathbb{R}^n \times (0,+\infty)$.

Exercise 4.2 (Maybe wait next week for this one)**.** Prove the following Lemma 4.1.

Lemma 4.1. *Let $T > 0$ and $g \in C^0(\mathbb{R}^n \times [0,T]) \cap L^\infty(\mathbb{R}^n \times [0,T])$. Then, for every $\hat{x} \in \mathbb{R}^n$,*

$$\lim_{\substack{(x,t)\to(\hat{x},0) \\ t>0}} \int_{\mathbb{R}^n} \Phi(z,t)g(\hat{x}-z,t)\,\mathrm{d}z = g(\hat{x},0). \tag{9}$$