# Introduction to Partial Differential Equations – Exercises for Week 2 –

## Contents



## 1. Spherical averages

Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open sets. For $u \in L^1_{\mathrm{loc}}(X \times Y)$, $x \in X$, $y \in Y$ and $r > 0$ with $B(y,r) \subset Y$, define

$$\begin{aligned} \phi_u(x,y;r) &:= \fint_{B(y,r)} u(x,z)\,\mathrm{d}z = \fint_{B_Y(0,1)} u(x,y+rz)\,\mathrm{d}z, \\ \psi_u(x,y;r) &:= \fint_{\partial B(y,r)} u(x,z)\,\mathrm{d}S(z) = \fint_{\partial B(0,1)} u(x,y+rz)\,\mathrm{d}S(z). \end{aligned} \tag{1}$$

**Exercise 1.1.** Show that, if $u$ is continuous, then

$$u(x,y) = \lim_{r\to 0} \phi_u(x,y;r) = \lim_{r \to 0} \psi_u(x,y;r). \tag{2}$$

**Exercise 1.2.** Show the following statements.

If $u \in C^{a;b}(X \times Y)$, with $a, b \in \mathbb{N}$, then, for every $\epsilon > 0$, $\phi_u, \psi_u \in C^{a;b;\lfloor b/2 \rfloor}(X \times Y_\epsilon \times (0,\epsilon))$, where $Y_\epsilon = \{y \in Y : d(y, \partial Y) > \epsilon\}$.

Moreover, for all $\alpha \in \mathbb{N}^m$ and $\beta \in \mathbb{N}^n$, with $|\alpha| \le a$ and $|\beta| \le b$,

$$D^\alpha_x D^\beta_y \phi_u = \phi_{D^\alpha_x D^\beta_y u} \qquad \text{and} \qquad D^\alpha_x D^\beta_y \psi_u = \psi_{D^\alpha_x D^\beta_y u}. \tag{3}$$

Finally,

$$\begin{aligned} &\text{if } b \ge 0, \quad \partial_r \phi_u(x,y;r) = \frac{n}{r}(\psi_u(x,y;r) - \phi_u(x,y;r)), \\ &\text{if } b \ge 2, \quad \partial_r \psi_u(x,y;r) = \frac{r}{n}\phi_{\triangle_y u}(x,y;r) \overset{(3)}{=} \frac{r}{n}\triangle_y \phi_u(x,y;r). \end{aligned} \tag{4}$$

February 26, 2025. Last git commit: 1d5f5ec in branch: master

## 2. Mean value formulas

**Exercise 2.1.** For all $n \ge 2$, compute

$$\psi_\Phi(0;r) := \fint_{\partial B(0,r)} \Phi(y)\,\mathrm{d}S(y),$$

where $\Phi$ is the fundamental solution of the Laplace equation. Is $r \mapsto \psi_\Phi(0;r)$ constant? Do we have a contradiction with the mean value formula proved in class?

**Exercise 2.2.** Show that, if $U \subset \mathbb{R}^n$ is open and $u \in C^0(U)$, then the following statements are equivalent:

(i) $\forall x \in U$, $\forall r > 0$, such that $\bar{B}(x,r) \subset U$,

$$u(x) = \fint_{\partial B(x,r)} u(y)\mathrm{d}S(y). \tag{5}$$

(ii) $\forall x \in U$, $\forall r > 0$, such that $\bar{B}(x,r) \subset U$,

$$u(x) = \fint_{B(x,r)} u(y)\mathrm{d}S(y). \tag{6}$$

## 3. Maximum principle

**Exercise 3.1.** Show that, if $u \in C^0(B(0,r);\mathbb{R})$ is such that $u \le M$ on $B(0,r)$, then $\fint_{B(x,r)} u(y)\,\mathrm{d}y \le M$, with equality if and only if $u = M$ on $B(0,r)$.

**Exercise 3.2.** Show the following statement: *Let $X$ be a locally connected topological space and $U \subset X$ open. Let $C \subset U$ be a connected component of $U$. Then $\partial C \subset \partial U$.*

After this, show that the hypothesis of $X$ being locally connected is necessary. In other words, find an example of a topological space $X$ that is not locally connected, and $U \subset X$ open that has a connected component $C \subset U$ with $\partial C \cap \partial U = \emptyset$.

**Exercise 3.3.** State and prove the *Strong Minimum Principle* for harmonic functions, both in the first and second versions.

**Exercise 3.4.** Show the strong maximum and minimum principles for harmonic functions without using the mean value property, i.e., explicitly using that $\triangle u = 0$.

*Hint*: First consider $u \in C^2(U;\mathbb{R})$ with $\triangle u > 0$ in $U$. Suppose $x \in U$ is such that $u(x) = \max_U u$. Then $t = 0$ is a point of maximum for $t \mapsto u(x + te_j)$, for each $j \in \{1,\dots,n\}$. Therefore $\frac{\partial^2 u}{\partial x_j^2}(x) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} u(x+te_j) \le 0$.

**Exercise 3.5.** Write a proof of the following corollary.

**Corollary 3.1** (Monotonicity of Laplace's boundary value problem)**.** *Let $U \subset \mathbb{R}^n$ open, bounded and connected. Let $u \in C^2(U;\mathbb{R}) \cap C^0(\bar{U};\mathbb{R})$ and $g \in C^0(\partial U;\mathbb{R})$ be such that*

$$\begin{cases} \triangle u = 0 & \textit{in } U, \\ u = g & \textit{on } U. \end{cases}$$

*If $g$ is not constant and $g \ge 0$ on $\partial U$, then $u > 0$ on $U$.*

## 4. Subharmonic and superharmonic functions

Let $U \subset \mathbb{R}^n$ open. A function $u \in C^2(U)$ is *subharmonic* if and only if $\triangle u \geq 0$ on $U$.

**Exercise 4.1.** State and prove a theorem "Mean value formula for subharmonic functions".

[*Remark*: Notice that subharmonic do NOT have a mean-value property, but something similar: follow the proof we have done in class for harmonic functions and adapt it to subharmonic functions. You will get some statement.]

**Exercise 4.2.** State and prove a theorem "Strong maximum principle for subharmonic functions".

**Exercise 4.3.** Find examples of subharmonic functions that are not harmonic. Find a subharmonic function that does not satisfy the minimum principle.

A function $u \in C^2(U)$ is *superharmonic* if and only if $\triangle u \leq 0$ on $U$.

**Exercise 4.4.** Do Exercises 4.1, 4.2 and 4.3 for superharmonic functions, exchanging "maximum" and "minimum".

**Exercise 4.5.** Write and prove a version of Corollar **??** for sub- and superharmonic functions.