INTRODUCTION TO PARTIAL DIFFERENTIAL EQUATIONS –LECTURE NOTES–

SEBASTIANO NICOLUSSI GOLO

Contents

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Introduction

§0.1. Bibliography. I will closely follow [5]. Other references are:

- L. C. Evans. *Partial differential equations.* Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749
- G. B. Folland. *Introduction to partial differential equations.* Second. Princeton University Press, Princeton, NJ, 1995, pp. xii+324
- H. Brézis and F. Browder. "Partial differential equations in the 20th century". English. In: *Adv. Math.* 135.1 (1998), pp. 76–144

Part 0. Preliminaries

In this first preliminary part, we will touch superficially several notions that we will refer to along the lecture notes.

1. Pairings, dot and scalar products, etc...

There is a flourishing art of making products of stuff in mathematics and physics. But notation lacks. So, here are my choices.

§1.1. Dot Product. If $a, b \in \mathbb{C}^n$, then

$$a \cdot b = \sum_{j=1}^{n} a_j b_j.$$

§1.2. Scalar or inner product. If $a, b \in \mathbb{C}^n$, then

$$\langle a, b\rangle = \sum_{j=1}^{n} a_j \bar{b}_j = a \cdot \bar{b}.$$

More generally, if μ is a measure and $a, b \in L^2(\mu; \mathbb{C}^n)$ are complex-valued functions

$$\langle a, b\rangle = \langle a, b\rangle_\mu = \int a(x) \cdot \bar{b}(x)\,\mathrm{d}\mu(x).$$

These are inner products, with the properties ($\lambda \in \mathbb{C}$)

$$\langle a, b\rangle = \overline{\langle b, a\rangle}, \qquad \langle a, \lambda b\rangle = \bar{\lambda}\langle a, b\rangle,$$
$$\langle a, a\rangle \in [0, +\infty), \qquad \langle \lambda a, b\rangle = \lambda\langle a, b\rangle.$$

§1.3. Pairing. If V is a vector space (over some field $\mathbb{K}$, e.g., $\mathbb{K} = \mathbb{C}$) with algebraic dual V^* (that is, the space of all linear maps $V \to \mathbb{K}$), then, for all $a \in V^*$ and $b \in V$,

$$\langle a|b\rangle = a[b] \in \mathbb{K} \qquad \text{(which is the evaluation of } a \text{ in } b) .$$

Sometimes, we can make it more precise

$$_{V^*}\langle a|b\rangle_V = a[b].$$

In general, if $b \in V$ and $a \in \mathrm{Lin}(V; W)$ for some linear space W,

$$\langle a|b\rangle = a[b] \in W.$$

With this in mind, it is clear that

$$\langle a|b\rangle = \langle b|a\rangle.$$

For instance, $_{V^*}\langle a|b\rangle_V = {}_V\langle b|a\rangle_{V^*}$.

Pairings have the following properties ($\lambda \in \mathbb{K}$):

$$\langle a|b\rangle = \langle b|a\rangle,$$
$$\langle \lambda a|b\rangle = \lambda\langle a|b\rangle,$$
$$\langle a|\lambda b\rangle = \lambda\langle a|b\rangle.$$

§1.4. Example in $L^2(\mu)$. To put everything together, we see that, if $a, b \in L^2(\mu)$, define $B, \bar{B} \in L^2(\mu)^*$ as

$$B[\phi] = \int b \cdot \phi\,\mathrm{d}\mu, \qquad \forall \phi \in L^2(\mu),$$
$$\bar{B}[\phi] = \langle \phi, b\rangle, \qquad \forall \phi \in L^2(\mu).$$

Then

$$\langle a, b\rangle = \int a \cdot \bar{b}\,\mathrm{d}\mu = \langle \bar{B}|a\rangle = \overline{\langle B|\bar{a}\rangle}$$

and

$$\langle B|a\rangle = \langle a|\bar{b}\rangle = \langle b|\bar{a}\rangle.$$

2. Derivatives

Let $n, m \in \mathbb{N}$. We denote by $e_1, \dots, e_n$ the standard basis of $\mathbb{R}^n$.

Let $\Omega \subset \mathbb{R}^n$ be an open set and $f : \Omega \to \mathbb{C}^m$ and $x \in \Omega$. The derivative of f at x, if it exists, is the $\mathbb{R}$-linear map $\mathrm{D}f(x) : \mathbb{R}^n \to \mathbb{C}^m$ such that

$$\lim_{y \to x} \frac{\|f(y) - f(x) - \mathrm{D}f(x)[y - x]\|}{\|y - x\|} = 0.$$

The partial derivative of f are

$$\frac{\partial f}{\partial x_j}(x) := \mathrm{D}f(x)[e_j] = \lim_{h \to 0} \frac{f(x + he_j) - f(x)}{h} \in \mathbb{C}^m.$$

The gradient of f is (if it exists)

$$\nabla f := \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

The gradient is mostly used for scalar functions, that is, when $m = 1$.

If $\alpha \in \mathbb{N}^n$ is a multiindex, we set $|\alpha| := \sum_{j=1}^n \alpha_j$ and we denote by $\partial^\alpha f$ or $\mathrm{D}^\alpha f$ the partial derivative of f given by

$$\mathrm{D}^\alpha f = \left(\frac{\partial}{\partial x_1} \right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n} \right)^{\alpha_n} f.$$

If a function $f(x, y)$ depends on several variables, we write $\mathrm{D}_x f$ or $\partial_x f$ to denote the derivatives of f in the directions of x.

The space of functions $\Omega \to \mathbb{C}^m$ that are continuous, differentiable and with continuous derivatives up to order $k \in \mathbb{N}$ is denote by $C^k(\Omega; \mathbb{C}^m)$. If $m = 1$, then we just write $C^k(\Omega)$. We write $C^k(\bar{\Omega}; \mathbb{C}^m)$ for the space of functions that are smooth of class C^k on a neighborhood of $\bar{\Omega}$.

If the domain of a function u is described as a product of open sets $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, so $u : X \times Y \to \mathbb{C}$, we may require different regularity in the two entries. So, we say that $u \in C^{a;b}(X \times Y)$ for some $a, b \in \mathbb{N}$ if, for every $y \in Y$, $u(\cdot, y) \in C^a(X)$, and for every $x \in X$, $u(x, \cdot) \in C^b(Y)$.

Notice that $C^{a,b}(X)$ (with comma in place of semicolumn) is usually reserved for functions of class C^a whose a-th order derivative is b-Hölder. In this case, we would have $a \in \mathbb{N}$ and $b \in [0, 1]$.

3. Measures

§3.1. L^p spaces. Let $(X, \mathscr{M}, \mu)$ be a measure space, that is, X is a set endowed with a σ-algebra $\mathscr{M} \subset 2^X$ and $\mu : \mathscr{M} \to [0, +\infty]$ is a measure. In fact, we will mostly have $X = \Omega \subset \mathbb{R}^n$ open subset of $\mathbb{R}^n$, $\mathscr{M}$ the Borel σ-algebra and μ the Lebesgue measure.

For $u : X \to \mathbb{C}$ measurable and $p \in [1, +\infty)$, define

$$\|u\|_{L^p} := \|u\|_{L^p(X)} := \left(\int_X |u(x)|^p \,\mathrm{d}\mu(x) \right)^{1/p}.$$

The space $L^p(X)$ or $L^p(\mu)$ is the Banach space of all equivalence classes of measurable functions $u : X \to \mathbb{C}$ with $\|u\|_{L^p} < \infty$, where two functions are identified if they are equal μ-almost everywhere.

Sometimes, when we write "$u \in L^p(X)$", we mean that u is a specific function, not an equivalence class.

§3.2. Hölder and Minkowski inequalities. Let $(X, \mathscr{M}, \mu)$ be a measure space. Let $u, v : X \to \mathbb{C}$ and $u_j : X \to \mathbb{C}$, $j \in \{1, \dots, k\}$, be measurable functions.

(Hölder)
$$\forall p, p' \in [1, +\infty), \text{ with } \tfrac{1}{p} + \tfrac{1}{p'} = 1, \quad \int_X u(x) v(x) \,\mathrm{d}\mu(x) \le \|u\|_{L^p(X)} \|v\|_{L^{p'}(X)}.$$

(General Hölder)
$$\forall \{p_j\}_{j=1}^k \subset [1, +\infty), \text{ with } \textstyle\sum_{j=1}^k \frac{1}{p_j} = 1, \quad \int_X \prod_{j=1}^k u_j(x) \,\mathrm{d}\mu(x) \le \prod_{j=1}^k \|u_j\|_{L^{p_j}(X)}.$$

(Minkowski) $\forall p \in [1, \infty]$ $\|u + v\|_{L^p} \le \|u\|_{L^p(X)} + \|v\|_{L^p(X)}$.

§3.3. Approximation with continuous functions.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^n$ open and $p \in [1, \infty)$. The space $C_c(\Omega)$ of continuous functions with compact support in Ω is dense in $L^p(\Omega)$.*

Proof. [8, Proposition (7.9)] □

§3.4. Dominated Convergence Theorem.

Theorem 3.2 (LDCT, Lebesgue Dominated Convergence Theorem [8, Theorem 2.24])**.** *Let (X, μ) be a measure space, T a topological space with $\hat{t} \in T$, and $f : T \times X \to \mathbb{R}$ a function such that there exists $g \in L^1(\mu)$ with $|f(t,x)| \le g(x)$ for all $(t,x) \in T \times X$. Suppose that, for μ-a.e. $x \in X$, the limit $\lim_{t \to \hat{t}} f(t,x)$ exists. Then*

$$\lim_{t \to \hat{t}} \int_X f(t,x)\,\mathrm{d}\mu(x) = \int_X (\lim_{t \to \hat{t}} f(t,x))\,\mathrm{d}\mu(x).$$

§3.5. Functions defined by integrals.

Theorem 3.3 (Integral with parameter)**.** *Let $X \subset \mathbb{R}^m$ be an open set and (Y, μ) a measure space. Let $K : X \times Y \to \mathbb{C}$ be a measurable function and set*

$$F(x) := \int_Y K(x,y)\,\mathrm{d}\mu(y),$$

for all $x \in X$ such that $y \mapsto K(x,y)$ is μ-integrable, i.e., such that $\int_Y |K(x,y)|\,\mathrm{d}\mu(y) < \infty$.

(3.3.1) *If*

$$\exists g \in L^1(Y) \quad \forall (x,y) \in X \times Y \qquad |K(x,y)| \le g(y) \tag{1}$$

then F is defined for all $x \in X$ and $F \in L^\infty(X)$. In particular, $\|F\|_{L^\infty(X)} \le \|g\|_{L^1(Y)}$.

(3.3.2) *If* (1) *holds, and if*

$$\forall y \in Y \qquad K(\cdot, y) \in C^0(X), \tag{2}$$

then $F \in C^0(X)$.

(3.3.3) *If* (1) *and* (2) *hold, and if*

$$\forall y \in Y \;:\; K(\cdot,y) \in C^1(X), \text{ and } \exists g_1 \in L^1(Y) \forall (x,y) \in X \times Y \;:\; |D_x K(x,y)| \le g_1(y), \tag{3}$$

then $F \in C^1(X)$ and, for every $x \in X$,

$$D_x F(x) = \int_Y D_x K(x,y)\,\mathrm{d}\mu(y). \tag{4}$$

Proof. Proof of **(3.3.1)**. The condition (1) implies directly that $K(x, \cdot) \in L^1(Y)$ and that, for every $x \in X$,

$$|\int_Y K(x,y)\,\mathrm{d}\mu(y)| \le \int_Y |K(x,y)|\,\mathrm{d}\mu(y) \le \|g\|_{L^1(Y)}.$$

Proof of **(3.3.2)**. Let $\{x_j\}_{j \in \mathbb{N}} \subset X$ be a sequence converging to $x_\infty \in X$. Condition (1) allows us to apply the Lebesgue Dominated Convergence Theorem 3.2,

$$\begin{aligned} F(x_\infty) &= \int_Y K(x_\infty, y)\,\mathrm{d}\mu(y) \\ [\text{by } K(\cdot,y) \in C^0(X)] &= \int_Y \lim_{j \to \infty} K(x_j, y)\,\mathrm{d}\mu(y) \\ [\text{by LDCT and (1)}] &= \lim_{j \to \infty} \int_Y K(x_j,y)\,\mathrm{d}\mu(y) = \lim_{j \to \infty} F(x_j). \end{aligned}$$

This shows that $F \in C^0(X)$.

Proof of **(3.3.3)**. Fix $\hat{x} \in X$ and $i \in \{1, \dots, m\}$ and $\hat{h} > 0$ such that $B(\hat{x}, \hat{h}) \Subset X$. For $h \in (-\hat{h}, \hat{h})$, we have

$$\frac{F(\hat{x} + he_i) - F(\hat{x})}{h} = \int_Y \frac{K(\hat{x} + he_i, y) - K(\hat{x}, y)}{h}\,\mathrm{d}\mu(y).$$

Notice that for every $h \in (-\hat{h}, \hat{h})$ and every $y \in Y$ there exists $h' \in (-h, h)$ such that

$$\frac{K(\hat{x} + he_i, y) - K(\hat{x}, y)}{h} = \frac{\partial K}{\partial x_i}(\hat{x} + h' e_i, y).$$

Hence, by (3),

$$\left| \frac{K(\hat{x} + he_i, y) - K(\hat{x}, y)}{h} \right| \le \left| \frac{\partial K}{\partial x_i}(\hat{x} + h' e_i, y) \right| \le g_1(y). \tag{5}$$

Therefore, we can apply the Lebesgue Dominated Convergence Theorem 3.2, to obtain

$$\begin{aligned}
\int_Y \frac{\partial K}{\partial x_i}(\hat{x}, y) \,\mathrm{d}y &= \int_Y \lim_{h \to 0} \frac{K(\hat{x} + he_i, y) - K(\hat{x}, y)}{h} \,\mathrm{d}\mu(y) \\
\text{[by LDCT and (5)]} &= \lim_{h \to 0} \int_Y \frac{K(\hat{x} + he_i, y) - K(\hat{x}, y)}{h} \,\mathrm{d}\mu(y) \\
&= \lim_{h \to 0} \frac{F(\hat{x} + he_i) - F(\hat{x})}{h}.
\end{aligned}$$

This shows that F is differentiable at every point, with identity (4). By **(3.3.2)**, $D_x F$ is continuous. □

Corollary 3.4. *Let $X \subset \mathbb{R}^m$ be an open set and (Y, μ) a measure space. Let $K : X \times Y \to \mathbb{C}$ be a measurable function and set*

$$F(x) = \int_Y K(x, y) \,\mathrm{d}\mu(y),$$

for all $x \in X$ such that $y \mapsto K(x, y)$ is integrable.

Suppose that

$$\begin{gathered}
\forall y \in Y \;:\; K(\cdot, y) \in C^\infty(X), \text{ and} \\
\forall \alpha \in \mathbb{N}^m \;\exists g_\alpha \in L^1(Y) \;\forall (x, y) \in X \times Y \;:\; |D_x^\alpha K(x, y)| \le g_1(y).
\end{gathered}$$

Then $F \in C^\infty(X)$.

The following Proposition 3.5 is a direct consequence of Theorem 3.3 and the Hölder inequality.

Proposition 3.5. *Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open sets, $k \in \mathbb{N}$ and $p \in [1, \infty]$. Let $K : X \times Y \to \mathbb{C}$ be a measurable function such that:*

1. *for every $y \in Y$, the function $x \mapsto K(x, y)$ belongs to $C^k(Y)$;*
2. *for every $E \Subset X$ and $\alpha \in \mathbb{N}^n$ with $|\alpha| \le k$, there exists $g_{\alpha,E} \in L^p(Y)$ such that, for every $x \in E$ and every $y \in Y$,*

$$\left| \frac{\partial^{|\alpha|} K}{\partial x^\alpha}(x, y) \right| \le g_{\alpha,E}(y).$$

For $q \in [1, \infty]$ with $1/p + 1/q = 1$ and $f \in L^q(Y)$, define $T_K f$ as

$$T_K f(x) = \int_Y K(x, y) f(y) \,\mathrm{d}y.$$

Then $T_K f \in C^k(X)$ and, for every $\alpha \in \mathbb{N}^m$ with $|\alpha| \le k$,

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} T_K f = T_{\frac{\partial^{|\alpha|} K}{\partial x^\alpha}} f.$$

§3.6. Fundamental Theorem of Calculus of Variations.

Theorem 3.6 (Fundamental Theorem of Calculus of Variations)**.** *If $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is such that*

$$\forall \phi \in C_c^\infty(U), \qquad \int_U \phi \cdot u \,\mathrm{d}x = 0,$$

then $u = 0$ almost everywhere in $\mathbb{R}^n$.

Exercise 3.7. Prove the Fundamental Theorem of Calculus of Variations 3.6. ♢

4. Divergence theorem

Given an open set $U \subset \mathbb{R}^n$, the divergence of a vector field $v \in C^1(U;\mathbb{C}^n)$ is

$$\operatorname{div}(v) = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \in C(U).$$

Theorem 4.1 (Divergence Theorem [14, Lemma 10.1]). *Let $U \subset \mathbb{R}^n$ be an open set with C^1 boundary ∂U. We denote by $\nu : \partial U \to \mathbb{S}^{n-1}$ the outward normal. Let $v \in C^1(\bar{U};\mathbb{C}^n)$ be a vector field. Then*

$$\int_U \operatorname{div}(v)\,\mathrm{d}x = \int_{\partial U} v \cdot \nu \,\mathrm{d}S(x). \tag{6}$$

Recall that the integral in $\mathrm{d}S$ is the surface integral over ∂U. One can think of $\mathrm{d}S$ as the Hausdorff measure of dimension $n-1$.

Remark 4.2. You might have seen the divergence Theorem 4.1 for real vector fields. In the case of a complex vector field, we have $v = v_1 + iv_2$ with $v_1, v_2 \in C^1(\bar{U};\mathbb{R}^n)$. Then

$$\begin{aligned}\int_U \operatorname{div}(v)\,\mathrm{d}x &= \int_U \operatorname{div}(v_1)\,\mathrm{d}x + i\int_U \operatorname{div}(v_2)\,\mathrm{d}x\\ &= \int_{\partial U} v_1 \cdot \nu\,\mathrm{d}S(x) + i\int_{\partial U} v_2\cdot \nu\,\mathrm{d}S(x)\\ &= \int_{\partial U} (v_1 + iv_2)\cdot \nu\,\mathrm{d}S(x).\end{aligned}$$

Notice that ν has necessarily real components and thus $\bar{\nu} = \nu$. In other words, we could also write $v \cdot \nu = v \cdot \bar{\nu} = \langle v, \nu\rangle$, as the hermitian product of complex vectors.

Remark 4.3. Formula (6) is usually paired with the following formula for the divergence: if $f \in C^1(U)$ and $v \in C^1(U;\mathbb{C}^n)$, then

$$\operatorname{div}(fv) = \nabla f \cdot v + f\operatorname{div}(v). \tag{7}$$

Indeed,

$$\begin{aligned}\operatorname{div}(fv) &= \sum_{j=1}^{n} \frac{\partial (f v_j)}{\partial x_j}\\ &= \sum_{j=1}^{n}\left(\frac{\partial f}{\partial x_j} v_j + f\frac{\partial v_j}{\partial x_j}\right)\\ &= \nabla f \cdot v + f\operatorname{div}(v).\end{aligned}$$

It follows that

$$\int_U (\nabla f \cdot v + f\operatorname{div}(v))\,\mathrm{d}x = \int_{\partial U} f v\cdot\nu\,\mathrm{d}S(x).$$

5. Coarea formulas

§5.1. Intro to surface measures. If $\Sigma \subset \mathbb{R}^n$ is an m-dimensional immersed C^1-submanifold, we denote by S^m the m-dimensional surface measure on Σ. We can describe S^m as the Hausdorff m-dimensional measure, or, being the submanifold smooth, as an integral of m-differential forms on Σ. We can also obtain several formulas for its explicit use. However, we will use a list of properties of these measures, and we do not need further details.

We need the symmetries of S^m: it is translation invariant, rotation invariant and scales properly under dilations. More precisely, if $\Sigma \subset \mathbb{R}^m$ is an immersed submanifold and if $O \in \mathsf{O}(n)$ is an orthogonal matrix, $v \in \mathbb{R}^n$ and $r > 0$, then

$$\int_{r(O\Sigma+v)} u(x)\mathrm{d}S^m(x) = r^m \int_\Sigma u(r(Ox+v))\mathrm{d}S^m(x), \qquad \forall u \in C^0(\mathbb{R}^n). \tag{8}$$

§5.2. Coarea formula.

Theorem 5.1 (Coarea Formula:[2, Theorem 2.93& Remark 2.94])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $F : \Omega \to \mathbb{R}^k$ a C^1-submersion, that is, a C^1-smooth map with surjective differential at each point. As a consequence, we have that $F(\Omega)$ is open in $\mathbb{R}^k$ and that, for every $y \in F(\Omega)$, the set $F^{-1}(y) \subset \Omega$ is an immersed submanifold of dimension $n-k$.*

Then, for every $u \in L^1(\Omega)$ with compact support,

$$\int_\Omega u(x)J(DF(x))\,\mathrm{d}x = \int_{F(\Omega)}\int_{F^{-1}(y)} u(x)\mathrm{d}S^{n-k}(x)\mathrm{d}y, \tag{9}$$

where

$$J(DF(x)) = \sqrt{\det(DF(x)\times DF(x)^T)} = \sqrt{\sum_{B\in\{k\times k \text{ minors of } DF(x)\}} \det(B)^2}. \tag{10}$$

Exercise 5.2. Compute $J(DF)$ as in (10) when $k=1$ and when $k=n-1$. ♢

§5.3. Consequences of the coarea formula: spherical integrals.

Proposition 5.3. *If $\Omega \subset \mathbb{R}^n$ is open, then*

$$\int_\Omega u(x)\,\mathrm{d}x = \int_0^\infty \int_{\partial B(0,r)\cap\Omega} u(x)\,\mathrm{d}S^{n-1}(x)\,\mathrm{d}r \qquad \forall u \in C^0(\Omega). \tag{11}$$

Proof. Consider the function $F : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}$, $F(x) = |x|$. Then $J(DF(x)) = |\nabla F(x)| = 1$ for all $x \in \mathbb{R}^n \setminus \{0\}$. So, Coarea Formula (9) gives immediately the identity (11). □

§5.4. Consequences of the coarea formula: on the sphere. From Theorem 5.1, we can deduce seemingly more general results. For instance, a coarea formula on the sphere:

Proposition 5.4. *Let $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ centered at 0. If $f \in C^1(\mathbb{R}^n;\mathbb{R})$, then, for every $u \in C^0(\mathbb{S}^{n-1})$,*

$$\int_{\mathbb{S}^{n-1}} u(x)|\nabla f(x) - (\nabla f(x)\cdot x)x|\,\mathrm{d}S^{n-1}(x) = \int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}\cap\{f=z\}} u(x)\,\mathrm{d}S^{n-2}(x)\,\mathrm{d}z.$$

For example, if $f(x) = x_n$, then, for every $u \in C^0(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} u(x)\sqrt{1-x_n^2}\,\mathrm{d}S^{n-1}(x) = \int_{-1}^1\int_{\mathbb{S}^{n-1}\cap\{f=z\}} u(x)\,\mathrm{d}S^{n-2}(x)\,\mathrm{d}z. \tag{12}$$

Moreover, we have

$$\int_{\mathbb{S}^{n-1}} u(x)\,\mathrm{d}S^{n-1}(x) = \int_{-1}^1\int_{\mathbb{S}^{n-1}\cap\{x_3=z\}} u(x)\,\mathrm{d}S^{n-2}(x)\frac{1}{\sqrt{1-z^2}}\,\mathrm{d}z, \tag{13}$$

and

$$\int_{\partial B(0,r)} u(x)\,\mathrm{d}S^{n-1}(x) = \int_{-r}^r\int_{\partial B(0,r)\cap\{x_3=z\}} u(x)\,\mathrm{d}S^{n-2}(x)\frac{1}{\sqrt{r^2-z^2}}\,\mathrm{d}z. \tag{14}$$

Proof. Fix $u \in C^0(\mathbb{S}^{n-1})$ and $\epsilon > 0$. Define $\Omega_\epsilon = B(0,1+\epsilon)\setminus B(0,1)$ and $\tilde u \in C^0(\Omega_\epsilon)$ by $\tilde u(x) = u(x/|x|)$.

Define $F : \mathbb{R}^n\setminus\{0\} \to \mathbb{R}^2$ by $F(x) = (|x|, f(x))$. Notice that

$$\begin{aligned}
DF(x) &= \begin{pmatrix} \cdots & x/|x| & \cdots \\ \cdots & \nabla f(x) & \cdots \end{pmatrix},\\
DF(x)DF(x)^T &= \begin{pmatrix} 1 & \frac{x}{|x|}\cdot\nabla f(x) \\ \frac{x}{|x|}\cdot\nabla f(x) & \nabla f(x)\cdot\nabla f(x) \end{pmatrix},\\
\det(DF(x)DF(x)^T) &= |\nabla f(x)|^2 - \left(\frac{x}{|x|}\cdot\nabla f(x)\right)^2\\
&= \left|\nabla f(x) - \left(\frac{x}{|x|}\cdot\nabla f(x)\right)\frac{x}{|x|}\right|^2.
\end{aligned}$$

On the one hand, using the coarea formula for F, we have

$$\int_{\Omega_\epsilon} \tilde u(x)J(DF(x))\,\mathrm{d}x \overset{(9)}{=} \int_{F(\Omega_\epsilon)}\int_{\{F(x)=y\}} \tilde u(x)\,\mathrm{d}S^{n-2}(x)\,\mathrm{d}y$$

$$= \int_1^{1+\epsilon} \int_{\mathbb{R}} \int_{\{|x|=r, f(x)=z\}} u(x/r)\, \mathrm{d}S^{n-2}(x)\, \mathrm{d}z\, \mathrm{d}r.$$

On the other hand, using the coarea formula for spheres, that is (11),

$$\begin{aligned}\int_{\Omega_\epsilon} \tilde{u}(x) J(DF(x))\, \mathrm{d}x &\overset{(11)}{=} \int_1^{1+\epsilon} \int_{\partial B(0,r)} \tilde{u}(x) J(DF(x))\, \mathrm{d}S^{n-1}(x)\, \mathrm{d}r \\ &\overset{(8)}{=} \int_1^{1+\epsilon} r^{n-1} \int_{\partial B(0,1)} u(x) J(DF(rx))\, \mathrm{d}S^{n-1}(x)\, \mathrm{d}r.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} u(x) |\nabla f(x) - (\nabla f(x)\cdot x) x|\, \mathrm{d}S^{n-1}(x) \\ &\quad = \lim_{\epsilon\to 0} \fint_1^{1+\epsilon} r^{n-1} \int_{\partial B(0,1)} u(x) J(DF(rx))\, \mathrm{d}S^{n-1}(x)\, \mathrm{d}r \\ &\quad = \lim_{\epsilon\to 0} \fint_1^{1+\epsilon} \int_{\mathbb{R}} \int_{\{|x|=r, f(x)=z\}} u(x/r)\, \mathrm{d}S^{n-2}(x)\, \mathrm{d}z\, \mathrm{d}r \\ &\quad = \int_{\mathbb{R}} \int_{\mathbb{S}^{n-1}\cap\{f=z\}} u(x)\, \mathrm{d}S^{n-2}(x)\, \mathrm{d}z.\end{aligned}$$

Formula (12) follows from a direct computation.

The subsequent formula (13) is instead the result of a limit. For $u \in C^0(\mathbb{S}^{n-1})$, set

$$u_\epsilon(x) := \frac{\min\{1, (1-x_n^2)/\epsilon\}}{\sqrt{1-x_3^2}} \cdot u(x).$$

Then we plug u_ϵ into (12). On the one hand, we have

$$\begin{aligned}&\int_{\mathbb{S}^{n-1}} u_\epsilon(x) \sqrt{1-x_n^2}\, \mathrm{d}S^{n-1}(x) \\ &= \int_{\mathbb{S}^{n-1}\cap\{1-x_n^2\geq\epsilon\}} u(x)\, \mathrm{d}S^{n-1}(x) + \int_{\mathbb{S}^{n-1}\cap\{1-x_n^2<\epsilon\}} u(x) \frac{1-x_n^2}{\epsilon}\, \mathrm{d}S^{n-1}(x).\end{aligned}$$

If we take the limit $\epsilon \to 0$ we obtain

$$\lim_{\epsilon\to 0} \int_{\mathbb{S}^{n-1}\cap\{1-x_n^2\geq\epsilon\}} u(x)\, \mathrm{d}S^{n-1}(x) = \int_{\mathbb{S}^{n-1}} u(x)\, \mathrm{d}S^{n-1}(x)$$

and

$$\lim_{\epsilon\to 0} \int_{\mathbb{S}^{n-1}\cap\{1-x_n^2<\epsilon\}} u(x) \frac{1-x_n^2}{\epsilon}\, \mathrm{d}S^{n-1}(x) = 0.$$

On the other hand, we have

$$\begin{aligned}&\int_{-1}^{1} \int_{\mathbb{S}^{n-1}\cap\{x_3=z\}} u_\epsilon(x)\, \mathrm{d}S^{n-2}(x)\, \mathrm{d}z \\ &\quad = \int_{-\sqrt{1-\epsilon}}^{\sqrt{1-\epsilon}} \int_{\mathbb{S}^{n-1}\cap\{x_3=z\}} \frac{u(x)}{\sqrt{1-x_n^2}}\, \mathrm{d}S^{n-2}(x)\, \mathrm{d}z \\ &\qquad + \int_{[-1,1]\setminus[-\sqrt{1-\epsilon},\sqrt{1-\epsilon}]} \int_{\mathbb{S}^{n-1}\cap\{x_3=z\}} u(x) \frac{\sqrt{1-z^2}}{\epsilon}\, \mathrm{d}S^{n-2}(x)\, \mathrm{d}z.\end{aligned}$$

Taking the limit $\epsilon \to 0$, we obtain the right-hand side of (13).

The last formula (14) follows from the previous (13): take $\tilde{u}(x) := u(rx)$. Then

$$\begin{aligned}\int_{r\partial B(0,r)} u(x)\, \mathrm{d}S^{n-1}(x) &= r^{n-1} \int_{\partial B(0,1)} u(rx)\, \mathrm{d}S^{n-1}(x) \\ &\overset{(13)}{=} r^{n-1} \int_{-1}^{1} \int_{\mathbb{S}^{n-1}\cap\{x_3=z\}} u(rx)\, \mathrm{d}S^{n-2}(x) \frac{1}{\sqrt{1-z^2}}\, \mathrm{d}z \\ &\overset{\bar{z}=rz}{=} r^{n-1} \int_{-r}^{r} \int_{\mathbb{S}^{n-1}\cap\{x_3=\bar{z}/r\}} u(rx)\, \mathrm{d}S^{n-2}(x) \frac{1}{\sqrt{1-(\bar{z}/r)^2}} \frac{\mathrm{d}\bar{z}}{r} \\ &\overset{\bar{x}=rx}{=} \int_{-r}^{r} \int_{r\mathbb{S}^{n-1}\cap\{\bar{x}_3=\bar{z}\}} u(\bar{x})\, \mathrm{d}S^{n-2}(\bar{x}) \frac{r}{\sqrt{r^2-\bar{z}^2}} \frac{\mathrm{d}\bar{z}}{r}\end{aligned}$$

$$= \int_{-r}^{r} \int_{r\mathbb{S}^{n-1}\cap\{\bar{x}_3=\bar{z}\}} u(\bar{x})\,\mathrm{d}S^{n-2}(\bar{x}) \frac{1}{\sqrt{r^2-\bar{z}^2}}\,\mathrm{d}\bar{z}.$$

□

§5.5. Change of variables.

Theorem 5.5 (Change of variables). *Let $X, Y \subset \mathbb{R}^n$ be open sets and $\phi : X \to Y$ a diffeomorphism. Then, for every $u \in L^1(Y)$,*

$$\int_X u(\phi(x))|J\phi(x)|\,\mathrm{d}x = \int_Y u(y)\,\mathrm{d}y,$$

where $J\phi(x) = \det(D\phi(x))$.

Theorem 5.5 holds for the *non-oriented* integral. When dealing with oriented integrals, we need to take care of the sign of the jacobian $J\phi$. For example, line integrals are usually oriented: If $a < b$ and $f \in L^1([a, b])$, then $\int_a^b f(s)\,\mathrm{d}s$ is a *oriented* integral, while $\int_{[a,b]} f(s)\,\mathrm{d}s$ is *not oriented*. This distinction becomes clear by the identities

$$\int_{[a,b]} f(s)\,\mathrm{d}s = \int_a^b f(s)\,\mathrm{d}s = -\int_b^a f(s)\,\mathrm{d}s. \tag{15}$$

The root of the distinction is the following. A non-oriented integral is the integral of a function f over a measure space (X, μ): $\int_X f\,\mathrm{d}\mu$. A oriented integral is the integral of a differential form: for example, $\int_{\mathbb{R}^2} f(x, y)\mathrm{d}x \wedge \mathrm{d}y = -\int_{\mathbb{R}^2} f(x, y)\mathrm{d}y \wedge \mathrm{d}x$. In the case of integrals of differential forms, the change of variables states: $\int \phi^*\omega = \int \omega$ and $\phi^*(\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n) = J\phi \cdot (\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n)$.

On $\mathbb{R}^n$ with $n > 1$, we usually think of "dx" as $\mathrm{d}\mathcal{L}^n(x)$, where $\mathcal{L}^n$ is the Lebesgue measure, and not as the volume form $\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. On $\mathbb{R}$, instead, we usually think of "dx" as a 1-form.

Forgetting the details, the punchline is that, on line integrals, we need to keep in mind the indetities (15).

6. Spherical averages (or means)

Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be open sets. For $u \in L^1_{\mathrm{loc}}(X \times Y)$, $x \in X$, $y \in Y$ and $r > 0$ with $B(y, r) \subset Y$, define

$$\begin{aligned} \phi_u(x, y; r) &:= \fint_{B(y,r)} u(x, z)\,\mathrm{d}z = \fint_{B_Y(0,1)} u(x, y + rz)\,\mathrm{d}z, \\ \psi_u(x, y; r) &:= \fint_{\partial B(y,r)} u(x, z)\,\mathrm{d}S(z) = \fint_{\partial B(0,1)} u(x, y + rz)\,\mathrm{d}S(z). \end{aligned} \tag{16}$$

If u is continuous, it is clear that

$$u(x, y) = \lim_{r\to 0} \phi_u(x, y; r) = \lim_{r\to 0} \psi_u(x, y; r). \tag{17}$$

Lemma 6.1. *If $u \in C^{a;b}(X \times Y)$, with $a, b \in \mathbb{N}$, then, for every $\epsilon > 0$, $\phi_u, \psi_u \in C^{a;b;\lfloor b/2\rfloor}(X \times Y_\epsilon \times (0, \epsilon))$, where $Y_\epsilon = \{y \in Y : d(y, \partial Y) > \epsilon\}$.*

Moreover, for all $\alpha \in \mathbb{N}^m$ and $\beta \in \mathbb{N}^n$, with $|\alpha| \le a$ and $|\beta| \le b$,

$$D^\alpha_x D^\beta_y \phi_u = \phi_{D^\alpha_x D^\beta_y u} \qquad \text{and} \qquad D^\alpha_x D^\beta_y \psi_u = \psi_{D^\alpha_x D^\beta_y u}. \tag{18}$$

and

$$\begin{aligned} \partial_r \phi_u(x, y; r) &= \frac{n}{r}(\psi_u(x, y; r) - \phi_u(x, y; r)), \\ \partial_r \psi_u(x, y; r) &= \frac{r}{n}\phi_{\triangle_y u}(x, y; r) \overset{(18)}{=} \frac{r}{n}\triangle_y \phi_u(x, y; r). \end{aligned} \tag{19}$$

Proof. Notice that both definitions of ϕ_u and ψ_u falls into the framework of Theorem 3.3, which then implies (18). Moreover,

$$\begin{aligned} \phi_u(x, y; r) &= \frac{1}{\alpha_n r^n} \int_{B(y,r)} u(x, z)\,\mathrm{d}z \\ &= \frac{1}{\alpha_n r^n} \int_0^r \int_{\partial B_Y(y,s)} u(x, z)\,\mathrm{d}S(z)\,\mathrm{d}s \end{aligned}$$

$$= nr^{-n} \int_0^r s^{n-1} \psi_u(x, y; s) \, \mathrm{d}s.$$

Therefore,

$$\begin{aligned} \partial_r \phi_u(x, y; r) &= -n^2 r^{-n-1} \int_0^r s^{n-1} \psi(x, y; s) \, \mathrm{d}s + nr^{-1} \psi_u(x, y; r) \\ &= \frac{n}{r}(-\phi_u(x, y; r) + \psi_u(x, y; r)). \end{aligned}$$

Finally,

$$\begin{aligned} \partial_r \psi_u(x, y; r) &= \frac{1}{n\alpha_n} \partial_r \int_{\partial B(0,1)} u(x, y + rz) \, \mathrm{d}S(z) \\ \text{[by Theorem 3.3]} &= \frac{1}{n\alpha_n} \int_{\partial B(0,1)} (D_y u)(x, y + rz) \cdot z \, \mathrm{d}S(z) \\ \text{[by Divergence Theorem and (20)]} &= \frac{r}{n\alpha_n} \int_{B(0,1)} (\triangle_y u)(x, y + rz) \, \mathrm{d}z \\ &= \frac{r}{n} \phi_{\triangle_y u}(x, y; r), \end{aligned}$$

where

$$\operatorname{div}_z((D_y u)(x, y + rz)) = r(\operatorname{div}_y(D_y u))(x, y + rz) = r(\triangle_y u)(x, y + rz). \tag{20}$$

□

7. Change of coordinates

§7.1. Differential operator. Let $\Omega \subset \mathbb{R}^n$ be open. A *linear differential operator with smooth coefficients* on Ω is a map $P : C^\infty(\Omega) \to C^\infty(\Omega)$ of the form

$$P\phi = \sum_{\alpha \in \mathbb{N}^n} P_\alpha \mathrm{D}^\alpha \phi,$$

for some $P_\alpha \in C^\infty(\Omega)$, all zero but for finitely many indices.

§7.2. An abstract overview. Let $\Omega_1, \Omega_2 \subset \mathbb{R}^n$ be open sets and $\Phi : \Omega_1 \to \Omega_2$ a diffeomorphism. We have pull-backs and push-forwards of both functions and differential operators:

- Pull-back of functions: $\Phi^* : C^\infty(\Omega_2) \to C^\infty(\Omega_1)$, $\Phi^*\psi = \psi \circ \Phi$ for $\psi \in C^\infty(\Omega_2)$;
- Push-forward of functions: $\Phi_* : C^\infty(\Omega_1) \to C^\infty(\Omega_2)$, $\Phi_*\phi = \phi \circ \Phi^{-1}$ for $\psi \in C^\infty(\Omega_2)$;
- Push-forward of differential operators: if $P_1 : C^\infty(\Omega_1) \to C^\infty(\Omega_1)$ is a differential operator, then $\Phi_* P_1 = \Phi_* \circ P_1 \circ \Phi^*$;
- Pull-back of differential operators: if $P_2 : C^\infty(\Omega_2) \to C^\infty(\Omega_2)$ is a differential operator, then $\Phi^* P_2 = \Phi^* \circ P_2 \circ \Phi_*$.

The following diagrams might help:

$$\begin{array}{ccc} \Omega_1 & \xrightarrow{\Phi} & \Omega_2 \\ & & \\ C^\infty(\Omega_1) & \xleftarrow{\Phi^*} & C^\infty(\Omega_2) \\ \downarrow{\scriptstyle P_1} & & \downarrow{\scriptstyle \Phi_* P_1} \\ C^\infty(\Omega_1) & \xrightarrow{\Phi_*} & C^\infty(\Omega_2) \end{array} \qquad \begin{array}{ccc} \Omega_1 & \xrightarrow{\Phi} & \Omega_2 \\ & & \\ C^\infty(\Omega_1) & \xrightarrow{\Phi_*} & C^\infty(\Omega_2) \\ \downarrow{\scriptstyle \Phi^* P_2} & & \downarrow{\scriptstyle P_2} \\ C^\infty(\Omega_1) & \xleftarrow{\Phi^*} & C^\infty(\Omega_2) \end{array}$$

Notice that, for all functions $\phi \in C^\infty(\Omega_1)$, $\psi \in C^\infty(\Omega_2)$, and all differential operators $P_1 : C^\infty(\Omega_1) \to C^\infty(\Omega_1)$ and $P_2 : C^\infty(\Omega_2) \to C^\infty(\Omega_2)$,

$$\begin{aligned} \Phi_*\phi &= (\Phi^{-1})^*\phi = (\Phi^*)^{-1}\phi; \\ (\Phi_* P_1)\psi &= (P_1(\phi \circ \Phi)) \circ \Phi^{-1}; \\ (\Phi^* P_2)\phi &= (P_2(\phi \circ \Phi^{-1})) \circ \Phi; \\ \Phi_*\Phi^* &= \mathrm{Id}, \text{ and } \Phi^*\Phi_* = \mathrm{Id}. \end{aligned}$$

Notice that, if P_j, Q_j are differential operators, then

$$\Phi^*(P_2 \circ Q_2) = (\Phi^* P_2) \circ (\Phi^* Q_2), \text{ and } \Phi_*(P_1 \circ Q_1) = (\Phi_* P_1) \circ (\Phi_* Q_1). \tag{21}$$

Exercise 7.1. A differential operator of order zero on Ω is of the form $P\phi = f \cdot \phi$ for some $f \in C^\infty(\Omega)$. Compute $\Phi_* P$ and $\Phi^* P$ for differential operators P of order zero. Is it coherent with pull-back and push-forward of functions? ◊

Exercise 7.2. Let $\Phi : \Omega_1 \to \Omega_2$ be a diffeomorphism between open subsets of $\mathbb{R}^n$. For $j \in \{1, \dots, n\}$, compute $\Phi_* \partial_j$ and $\Phi^* \partial_j$. ◊

Exercise 7.3 (Harder)**.** Consider $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi(x) = Ax$ for $A \in \mathtt{GL}(\mathbb{R}^n)$, that is A is a $n \times n$ invertible matrix. For $\alpha \in \mathbb{N}^n$, compute $\Phi^* \mathrm{D}^\alpha$ and $\Phi_* \mathrm{D}^\alpha$. ◊

§7.3. An example: Laplacian in polar coordinates. Consider

$$\Phi : (0, +\infty) \times (-\pi, \pi) \to \mathbb{R}^2 \setminus (\{0\} \times [-\infty, 0]), \qquad \Phi(r, \theta) = (r\cos(\theta), r\sin(\theta)).$$

This function Φ is a *polar parametrization* of the plane: *polar coordinates* are in fact the inverse function of Φ. In other words, we can define functions

$$r, \theta : \mathbb{R}^2 \setminus (\{0\} \times [-\infty, 0]) \to \mathbb{R}, \text{ such that } \Phi(r(x, y), \theta(x, y)) = (x, y). \tag{22}$$

Anyway, with the function Φ above, we want to compute $\Phi^* \triangle$. To do this, using Exercise 7.2, we first compute

$$\mathrm{D}\Phi(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}, \qquad \mathrm{D}\Phi(r, \theta)^{-1} = \frac{1}{r} \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Therefore,

$$\begin{aligned} (\Phi^* \partial_x)(r, \theta) &= \mathrm{D}\Phi(r, \theta)^{-1} \left(\partial_x|_{\Phi(r,\theta)} \right) = \cos\theta \partial_r - \frac{\sin\theta}{r} \partial_\theta \\ (\Phi^* \partial_y)(r, \theta) &= \mathrm{D}\Phi(r, \theta)^{-1} \left(\partial_y|_{\Phi(r,\theta)} \right) = \sin\theta \partial_r + \frac{\cos\theta}{\partial r}. \end{aligned}$$

Using now (21), we obtain

$$\Phi^* \triangle = (\Phi^* \partial_x)^2 + (\Phi^* \partial_y)^2 = [...] = \partial_r^2 + \frac{1}{r^2} \partial_\theta^2 + \frac{1}{r} \partial_r. \tag{23}$$

This formula represents "the laplacian in polar coordinates".

§7.4. Another interpretation of the previous change of variables. We have another interpretation of the computation we made in (23). Define the vector fields

$$\begin{aligned} \vec{v}_r(x, y) &= \cos\theta \partial_x + \sin\theta \partial_y = \frac{x\partial_x - y\partial_y}{\sqrt{x^2 + y^2}}, \\ \vec{v}_\theta(x, y) &= r\sin\theta \partial_r + r\cos\theta \partial_y = -y\partial_x + x\partial_y, \end{aligned}$$

where we see θ and r as the functions defined in (22). These are the push-forward vector fields $\Phi_* \partial_r$ and $\Phi_* \partial_\theta$, respectively. As such, one usually simply writes ∂_r and ∂_θ for $\vec{v}_r$ and $\vec{v}_\theta$. We keep the distinction here for purely educational purposes.

At this point, we can write ∂_x and ∂_y in terms of $\vec{v}_r$ and $\vec{v}_\theta$:

$$\begin{aligned} \partial_x &= \cos\theta \vec{v}_r - \frac{\sin\theta}{r} \vec{v}_\theta, \\ \partial_y &= \sin\theta \vec{v}_r + \frac{\cos\theta}{r} \vec{v}_\theta. \end{aligned}$$

We can now perform the computation (23) again as

$$\triangle = \partial_x^2 + \partial_y^2 = (\cos\theta \vec{v}_r - \frac{\sin\theta}{r} \vec{v}_\theta)^2 + (\sin\theta \vec{v}_r + \frac{\cos\theta}{r} \vec{v}_\theta)^2 = [...] = \vec{v}_r^2 + \frac{1}{r^2} \vec{v}_\theta^2 + \frac{1}{r} \vec{v}_r,$$

where, we recall, r, θ are the functions defined in (22), and "$\vec{v}_r = \partial_r$", and "$\vec{v}_\theta = \partial_\theta$".

§7.5. Exercise: Laplacian in polar coordinates in arbitrary dimension.

Exercise 7.4. Consider the polar coordinates in $\mathbb{R}^n$ given by the function $\Phi : (0,+\infty)\times\mathbb{R}^{n-1}\to\mathbb{R}^n$,

$$\Phi(r,\theta_1,\dots,\theta_{n-1}) = r(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3,\dots,\sin\theta_1\cdots\sin\theta_{n-1}).$$

(1) Determine domain and image of Φ so that it becomes a diffeomorphism;
(2) Compute the laplacian in polar coordinates in $\mathbb{R}^n$.

$\Diamond$

8. Mollifiers

Let $\rho\in C_c^\infty(\mathbb{R}^n)$ be such that $\mathrm{spt}(\rho)=B(0,1)$, $0\le\rho\le 1$, $\rho(-x)=\rho(x)$ for all x, and $\int\rho\,\mathrm{d}x=1$. Define $\rho_\epsilon(x)=\rho(x/\epsilon)/\epsilon^n$. We call the family $\{\rho_\epsilon\}_{\epsilon>0}$ an *approximation of the identity on* $\mathbb{R}^n$, or *a family of mollifiers.*

For example, one can take

$$\rho(x)=\begin{cases} k\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1\\ 0 & \text{otherwise,}\end{cases} \tag{24}$$

where k normalizes the integral. The family of mollifiers given by the function ρ in (24) is called *standard family of mollifiers.*

Exercise 8.1. Show that the function

$$\rho_0(x)=\begin{cases} \exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1\\ 0 & \text{otherwise,}\end{cases}$$

is C^∞-smooth on $\mathbb{R}^n$ and compute $\int_{\mathbb{R}}^n\rho_0(x)\,\mathrm{d}x$. Show also that ρ_0 is not analytic. Does it exist a family of analytic mollifiers? $\Diamond$

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Proposition 8.2. $u*\rho_\epsilon\in C^\infty$ **TODO**

Part 1. Four classical PDE

9. Transport Equation

This is [5, Section 2.1]. The *transport equation* is the initial value problem

$$\begin{cases} \partial_t u + b \cdot \nabla u = f & \text{on } \mathbb{R}^n \times I, \\ u = g & \text{on } \mathbb{R}^n \times \{t_0\}, \end{cases} \tag{25}$$

where $I \subset \mathbb{R}$ is an interval, $t_0 \in I$, $b \in \mathbb{R}^n$, $f : \mathbb{R}^n \times I \to \mathbb{C}$ and $g : \mathbb{R}^n \to \mathbb{C}$. The function u is intended as a function in two variables, $u = u(x,t)$, where $x \in \mathbb{R}^n$ and $t \in I$. The derivative $\partial_t u$ is the derivative along the second variable, t, while the gradient ∇u is the derivative of u in the variable x.

If $f = 0$, then we call (25) the *homogeneous transport equation.* If $f \neq 0$, then (25) is the *nonhomogeneous transport equation.*

A solution to (25) is easily found.

Theorem 9.1. *Let $I \subset \mathbb{R}$ be an open interval, $t_0 \in \bar{I}$, $b \in \mathbb{R}^n$, $f \in C^0(\mathbb{R}^n \times \bar{I})$, and $g \in C^1(\mathbb{R}^n)$. Then the function $u \in C^1(\mathbb{R}^n \times \bar{I})$ defined by*

$$u(x,t) = g(x - (t - t_0)b) + \int_{t_0}^{t} f(x + (r - t)b, r) \, \mathrm{d}r \tag{26}$$

is the unique solution to (25).

Proof. To show that u is a solution to (25), we just compute the derivatives:

$$\begin{aligned} \partial_t u(x,t) &= -\nabla g(x - (t - t_0)b) \cdot b + f(x,t) - \int_{t_0}^{t} \nabla f(x + (r - t)b, r) \cdot b \, \mathrm{d}r, \\ \nabla u(x,t) &= \nabla g(x - (t - t_0)b) + \int_{t_0}^{t} \nabla f(x + (r - t)b, r) \, \mathrm{d}r. \end{aligned}$$

It follows that u solves (25).

To show that u is unique, suppose that $\tilde{u} \in C^1(\mathbb{R}^n \times \bar{I})$ is another solution. Then the difference $w := u - \tilde{u}$ solves (25) with $f = 0$ and $g = 0$. Notice that

$$\frac{\mathrm{d}}{\mathrm{d}s} w(x + sb, t + s) = \nabla w(x + sb, t + s) \cdot b + \partial_t w(x + sb, t + s) = 0$$

Therefore, for all $(x,t) \in \mathbb{R}^n \times I$, we have $w(x,t) = w(x + (t_0 - t)b, t_0) = 0$. □

Remark 9.2. The function u defined in (26) is well defined even in the case g is not smooth. In some sense, these other functions could be regarded as *weak solutions* to (25) when g is not C^1.

10. Laplace Equation

§10.1. Laplace operator. For $U \subset \mathbb{R}^n$ open and $u \in C^2(U)$, define the *laplacian of u* as

$$\triangle u := \operatorname{div}(\nabla u)) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}.$$

The operator $\triangle := \sum_{j=1}^{n} \partial_j^2$ is the *Laplace operator.*

§10.2. Harmonic function. A *harmonic function* on $U \subset \mathbb{R}^n$ open is a function $u \in C^2(U)$ such that $\triangle u = 0$.

Exercise 10.1. Show that the only harmonic functions on $\mathbb{R}$ are the affine functions. $\Diamond$

§10.3. Symmetries of Laplace operator. Let $U \subset \mathbb{R}^n$ open, $u \in C^2(U)$, $O \in \mathtt{O}(n)$, $b \in \mathbb{R}^n$, and $\lambda \in \mathbb{R} \setminus \{0\}$. Define $\bar{u}(y) := u(\lambda O y + b)$. Then $\bar{u} \in C^2(\lambda^{-1} O^{-1}(U - b))$ and

$$\triangle \bar{u}(y) = \lambda^2 (\triangle u)(\lambda O y + b).$$

§10.4. How to find the fundamental solution: radial solutions. In this section we solve the following Exercise 10.2, which asks to find radial harmonic functions on $\mathbb{R}^n$:

Exercise 10.2. For $n \geq 2$, find smooth non-constant functions $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{C}$ that are radial, that is, they only depend on the distance from the origin, and harmonic, that is, $\triangle u = 0$. $\Diamond$

In what follows, we solve Exercise 10.2. The reader is invited to solve it first by themselves: it is actually easier than it looks like at first.

Such radial harmonic functions are expected to exist because of the symmetries of the laplacian that we have seen in the previous Section **§10.3**: since the laplacian has spherical symmetry, we expect to have harmonic functions with spherical symmetry. We expect these functions to be somehow special: they are indeed, and we will see in the forthcoming sections that among them we find fundamental solutions of the laplacian.

Another type of symmetric harmonic functions are those that are homogeneous with respect to dilations. I invite the student to try to characterize those too: they are the so-called *spherical harmonics...*

Done with Exercise 10.2? Here is my take.

We consider functions of the form $u(x) = \phi(|x|^2)$, with $\phi : (0, +\infty) \to \mathbb{C}$ smooth. I choose to take the squared norm because in this way derivatives are easier. The we have

$$\begin{aligned} \frac{\partial u}{\partial x_j}(x) &= \phi'(|x|^2) 2x_j; \\ \frac{\partial^2 u}{\partial x_j^2}(x) &= \phi''(|x|^2) 4x_j^2 + \phi'(|x|^2) 2; \\ \triangle u(x) &= 4\phi''(|x|^2)\, |x|^2 + 2n\phi'(|x|^2). \end{aligned}$$

Since we want $\triangle u(x) = 0$ for $x \neq 0$, we obtain that ϕ must satisfy

$$\forall t \in (0, +\infty), \qquad 4\phi''(t) t + 2n\phi'(t) = 0. \tag{27}$$

Take $\psi = \phi'$: then ψ must satisfy

$$\forall t \in (0, +\infty), \qquad \frac{\psi'(t)}{\psi(t)} = -\frac{n}{2}\frac{1}{t}. \tag{28}$$

We are assuming $\psi(t) = \phi'(t) \neq 0$ for all $t > 0$, because if $\phi'(t) = 0$ for some t, then ϕ constant, i.e., $\phi' \equiv 0$, is a solution to the ODE (27). Since this ODE has unique solution given ϕ' at one point, we conclude that either ϕ is constant or ϕ' is never zero. So, ψ does satisfy (28).

The ODE (28) is equivalent to

$$\forall t \in (0, +\infty), \qquad \frac{\mathrm{d}}{\mathrm{d}t} \log(\psi(t)) = -\frac{n}{2} \frac{\mathrm{d}}{\mathrm{d}t} \log(t),$$

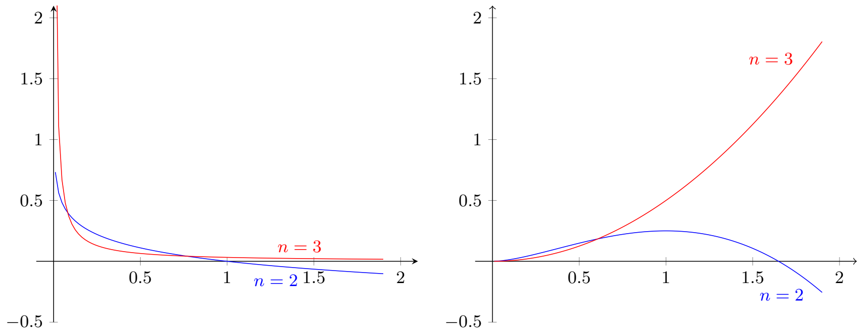


FIGURE 1. Plots of $|x| \mapsto \Phi(x)$ (left) and of $R \mapsto \int_{B(0,R)} \Phi(x)\,\mathrm{d}x$ (right) for $n = 2, 3$.

which is in turn equivalent to

$$\exists c \in \mathbb{C}, \quad \forall t \in (0, +\infty), \qquad \log(\psi(t)) = -\frac{n}{2}\log(t) + c.$$

Exponentiating and integrating, we obtain that non-constant solutions of (27) are

$$\exists a, c \in \mathbb{C} \quad \forall t \in (0, +\infty), \qquad \phi(t) = a + e^c \int_1^t s^{-\frac{n}{2}}\,\mathrm{d}s.$$

Here we need to distinguish tow cases:

$$n = 2: \qquad \phi(t) = a + e^c \log(t),$$

$$n > 2: \qquad \phi(t) = \left(a - \frac{2e^c}{2-n}\right) + \frac{2e^c}{2-n} t^{\frac{2-n}{2}}.$$

This might be pedantic, but we have shown that all functions required by Exercise 10.2 are all functions of the form

$$n = 2: \qquad u(x) = a + b\log(|x|),$$

$$n > 2: \qquad u(x) = a + \frac{b}{|x|^{n-2}},$$

for every choice of $a, b \in \mathbb{C}$.

§10.5. Fundamental solution of the Laplace's equation. Define $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi}\log(|x|) & \text{if } n = 2, \\ \frac{1}{n(n-2)\omega_n}\frac{1}{|x|^{n-2}} & \text{if } n \geq 3, \end{cases} \tag{29}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$. We will call this function Φ the *Fundamental solution of the Laplace's equation*. The most important property of this function is its role in Theorem **§10.6**. In fact, we will see fundamental solutions of linear operators from an abstract viewpoint in Section **§13.33**.

Proposition 10.3 (Properties of the fundamental solution)**.** *Let $n \geq 2$. The fundamental solution $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ of Laplace's equation, defined in* (29)*, satisfies the following statements.*

(1) Φ is an analytic function $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}$. If $n \geq 3$, then Φ is strictly positive valued.

(2) For every $x \in \mathbb{R}^n \setminus \{0\}$,

$$\nabla\Phi(x) = -\frac{1}{n\omega_n}\frac{x}{|x|^n}. \tag{30}$$

(3) For every $x \in \mathbb{R}^n \setminus \{0\}$,

$$\mathrm{D}^2\Phi(x) = \frac{1}{\omega_n}\frac{x \otimes x}{|x|^{n+2}} - \frac{1}{n\omega_n}\frac{\mathrm{Id}}{|x|^n}. \tag{31}$$

(4) $\triangle\Phi = 0$ *on* $\mathbb{R}^n \setminus \{0\}$.
(5) *There exists* $C > 0$ *such that for every* $x \in \mathbb{R}^n \setminus \{0\}$*:*
  (a) $0 < \Phi(x) \le C|x|^{2-n}$ *if* $n \ge 3$*;*
  (b) $\left|\frac{\partial \Phi}{\partial x_j}(x)\right| \le \frac{C}{|x|^{n-1}}$*, for all* $j \in \{1, \ldots, n\}$*;*
  (c) $\left|\frac{\partial^2 \Phi}{\partial x_j \partial x_k}(x)\right| \le \frac{C}{|x|^n}$*, for all* $j, k \in \{1, \ldots, n\}$*.*
(6) $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with*

$$\int_{B(0,R)} \Phi(x)\,\mathrm{d}x = \begin{cases} \frac{R^2}{4}(1 - 2\log(R)) & \text{if } n = 2, \\ \frac{R^2}{2(n-2)} & \text{if } n \ge 3. \end{cases} \tag{32}$$

See Figure **§10.4** *for a plot of these quantities.*

Proof. Part 1 is clear.

Statement 2 is proven as follows. If $n = 2$ and $x \in \mathbb{R}^2 \setminus \{0\}$, then

$$\nabla\Phi(x) = -\frac{1}{2\pi}\frac{1}{|x|}\frac{x}{|x|} = -\frac{1}{2\pi}\frac{x}{|x|^2}. \tag{33}$$

Therefore, $|\nabla\Phi(x)| = \frac{1}{2\pi}\frac{1}{|x|}$.

If $n \ge 3$ and $x \in \mathbb{R}^n \setminus \{0\}$, then

$$\nabla\Phi(x) = \frac{1}{n(n-2)\omega_n}(2-n)|x|^{2-n-1}\frac{x}{|x|} = -\frac{1}{n\omega_n}\frac{x}{|x|^n}. \tag{34}$$

Therefore, $|\nabla\Phi(x)| = \frac{1}{n\omega_n}\frac{1}{|x|^{n-1}}$. Notice that (33) and (34) are summarized in (30), because $\alpha(2) = \pi$.

For 3, we compute

$$\begin{aligned} \mathrm{D}^2\Phi(x) &= -\frac{1}{n\omega_n}\left(\frac{\mathrm{Id}}{|x|^n} + x \otimes \left(-n|x|^{-n-1}\frac{x}{|x|}\right)\right) \\ &= -\frac{1}{n\omega_n}\frac{\mathrm{Id}}{|x|^n} + \frac{1}{\omega_n}\frac{x \otimes x}{|x|^{n+2}}. \end{aligned}$$

In other words,

$$\begin{aligned} \frac{\partial^2\Phi}{\partial x_j \partial x_k}(x) &= \frac{\partial}{\partial x_j}\left(-\frac{1}{n\omega_n}\frac{x_k}{|x|^n}\right) \\ &= -\frac{1}{n\omega_n}\left(\frac{\delta_{jk}}{|x|^n} - x_k n|x|^{-n-1}\frac{1}{2}(\sum_\ell x_\ell^2)^{1/2} 2x_k\right) \\ &= -\frac{1}{n\omega_n}\frac{\delta_{jk}}{|x|^n} + \frac{1}{\omega_n}\frac{x_k x_j}{|x|^{n+2}}. \end{aligned}$$

So, we have (31).

Statement 4 is now an easy computation: if $x \in \mathbb{R}^n \setminus \{0\}$, then

$$\begin{aligned} \triangle\Phi(x) &= \mathrm{trace}(\mathrm{D}^2\Phi(x)) \\ &= \frac{1}{\omega_n}\frac{\mathrm{trace}(x \otimes x)}{|x|^{n+2}} - \frac{1}{n\omega_n}\frac{\mathrm{trace}(\mathrm{Id})}{|x|^n} \\ &= \frac{1}{\omega_n}\frac{1}{|x|^n} - \frac{1}{n\omega_n}\frac{n}{|x|^n} = 0. \end{aligned}$$

The estimates 5 are a consequence of the explicit formulas we have computed.

The last two formulas stated in part 6 are simply computed as follows. For $n = 2$, we have

$$\begin{aligned} \int_{B(0,R)} \Phi(x)\,\mathrm{d}x &= -\frac{1}{2\pi}\int_0^R \int_0^{2\pi} \log(r) r \,\mathrm{d}\theta\,\mathrm{d}r \\ &= -\int_0^R \log(r) r\,\mathrm{d}r \\ \text{[Integration by parts]} &= -\left(\log(r)\frac{r^2}{2}\Big|_0^R + \int_0^R \frac{1}{r}\frac{r^2}{2}\,\mathrm{d}r\right. \end{aligned}$$

$$= -\log(R)\frac{R^2}{2} + \frac{1}{2}\frac{R^2}{2} = \frac{R^2}{4}(1 - 2\log(R)).$$

For $n \geq 3$, we have

$$\begin{aligned}
\int_{B(0,R)} \Phi(x)\,\mathrm{d}x &= \frac{1}{n(n-2)\omega_n}\int_{B(0,R)} \frac{1}{|x|^{n-2}}\,\mathrm{d}x \\
&= \frac{1}{n(n-2)\omega_n}\int_0^R \int_{\partial B(0,1)} \frac{1}{|x|^{n-2}}\,\mathrm{d}S^{n-1}(x) r^{n-1}\,\mathrm{d}r \\
&= \frac{n\omega_n}{n(n-2)\omega_n}\int_0^R r\,\mathrm{d}r \\
&= \frac{1}{(n-2)}\frac{R^2}{2}.
\end{aligned}$$

□

§10.6. Solution to Poisson equation.

Theorem 10.4 (Solution to Poisson equation)**.** *Let $f \in C^2_c(\mathbb{R}^n)$ and define for $x \in \mathbb{R}^n$*

$$u(x) = \int_{\mathbb{R}^n} \Phi(y) f(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,\mathrm{d}x. \tag{35}$$

Then $u \in C^2(\mathbb{R}^n)$ and

$$-\triangle u = f \text{ in } \mathbb{R}^n. \tag{36}$$

Proof. Recall that, since $\Phi \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ by Proposition 10.3.6, and since f is continuous with bounded support, then, for every $x \in \mathbb{R}^n$, the integrand $y \mapsto \Phi(y)f(x-y)$ is integrable. It follows that u is well defined and that the identity between the second and the third expressions in (35) is a simple change of variables.

We need now to prove the regularity of u. For $x, y \in \mathbb{R}^n$, define

$$K_f(x,y) = \Phi(y) f(x-y).$$

Fix $R > 0$ and set $X = B(0,R)$. Let $S > 0$ be such that $\mathrm{spt}(f) \subset B(0,S)$: then, for every $x \in X$,

$$|K_f(x,y)| \leq \|f\|_{L^\infty} \cdot \Phi(y) \cdot \mathbb{1}_{B(0,R+S)}(y).$$

Notice that the function $g_f : y \mapsto \|f\|_{L^\infty} \cdot \Phi(y) \cdot \mathbb{1}_{B(0,R+S)}(y)$ is integrable over $\mathbb{R}^n$. By Theorem 3.3, the function u is continuous on X. Moreover, since

$$\mathrm{D}^\alpha_x K_f = K_{\mathrm{D}^\alpha f},$$

then, again by Theorem 3.3, $u \in C^2(X)$. Since this holds for every $R > 0$, we conclude that $u \in C^2(\mathbb{R}^n)$.

We have also obtained that, always from Theorem 3.3, for ever $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$,

$$\mathrm{D}^\alpha u(x) = \int_{\mathbb{R}^n} \Phi(y)(\mathrm{D}^\alpha f)(x-y)\,\mathrm{d}y.$$

We conclude that

$$\triangle u(x) = \int_{\mathbb{R}^n} \Phi(y)(\triangle f)(x-y)\,\mathrm{d}y.$$

We need to compute the latter integral.

For $\epsilon > 0$, we set

$$\triangle u(x) = \underbrace{\int_{B(0,\epsilon)} \Phi(y)(\triangle f)(x-y)\,\mathrm{d}y}_{I_\epsilon} + \underbrace{\int_{\mathbb{R}^n\setminus B(0,\epsilon)} \Phi(y)(\triangle f)(x-y)\,\mathrm{d}y}_{J_\epsilon}.$$

From (32) we obtain that, if $\epsilon < 1/2$, there is $C > 0$ (depending on n) such that

$$|I_\epsilon| \leq \|\triangle f\|_{L^\infty} \int_{B(0,\epsilon)} |\Phi(y)|\,\mathrm{d}y \leq \begin{cases} C\|\triangle f\|_{L^\infty}\epsilon^2|\log(\epsilon)| & \text{if } n = 2, \\ C\|\triangle f\|_{L^\infty}\epsilon^2 & \text{if } n \geq 3. \end{cases}$$

In both cases, we have

$$\lim_{\epsilon\to 0} I_\epsilon = 0.$$

For J_ϵ, we compute

$$
\begin{aligned}
J_\epsilon &= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y)(\triangle f)(x-y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \Phi(y)\triangle_y(f(x-y))\,\mathrm{d}y \\
&\overset{(7)}{=} \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \mathrm{div}_y(\Phi(y)\nabla_y f(x-y))\,\mathrm{d}y - \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \nabla_y \Phi(y)\cdot \nabla_y f(x-y)\,\mathrm{d}y \\
&\overset{(7)}{=} \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \mathrm{div}_y(\Phi(y)\nabla_y f(x-y))\,\mathrm{d}y + \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \mathrm{div}_y(\nabla_y \Phi(y))\cdot f(x-y)\,\mathrm{d}y \\
&\quad - \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \mathrm{div}_y(f(x-y)\nabla_y \Phi(y))\,\mathrm{d}y \\
&\overset{10.3.4}{=} \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \mathrm{div}_y(\Phi(y)\nabla_y f(x-y))\,\mathrm{d}y - \int_{\mathbb{R}^n \setminus B(0,\epsilon)} \mathrm{div}_y(f(x-y)\nabla_y \Phi(y))\,\mathrm{d}y \\
&\overset{(6)}{=} \underbrace{-\int_{\partial B(0,\epsilon)} \Phi(y)\nabla_y f(x-y)\cdot \frac{y}{|y|}\,\mathrm{d}S^{n-1}(y)}_{L_\epsilon} + \underbrace{\int_{\partial B(0,\epsilon)} f(x-y)\nabla_y \Phi(y)\cdot \frac{y}{|y|}\,\mathrm{d}S^{n-1}(y)}_{K_\epsilon}.
\end{aligned}
$$

We then have that, for $\epsilon < 1/2$, there is $C > 0$ (depending on n) such that

$$
\begin{aligned}
|L_\epsilon| &\le \|\nabla f\|_{L^\infty} \int_{\partial B(0,\epsilon)} \Phi(y)\,\mathrm{d}S^{n-1}(y) \\
&\le \begin{cases} C\|\nabla f\|_{L^\infty} |\log(\epsilon)|\epsilon & \text{if } n = 2, \\ C\|\nabla f\|_{L^\infty}\epsilon & \text{if } n \ge 3. \end{cases}
\end{aligned}
$$

In both cases, we have

$$\lim_{\epsilon\to 0} L_\epsilon = 0.$$

The remaining quantity is

$$
\begin{aligned}
K_\epsilon &= \int_{\partial B(0,\epsilon)} f(x-y)\nabla_y \Phi(y)\cdot \frac{y}{|y|}\,\mathrm{d}S^{n-1}(y) \\
&= -\frac{1}{n\omega_n}\int_{\partial B(0,\epsilon)} f(x-y)\frac{y}{|y|^n}\cdot \frac{y}{|y|}\,\mathrm{d}S^{n-1}(y) \\
&= -\frac{1}{n\omega_n}\int_{\partial B(0,\epsilon)} f(x-y)|y|^{2-n-1}\,\mathrm{d}S^{n-1}(y) \\
&= -\frac{1}{n\omega_n\epsilon^{n-1}}\int_{\partial B(0,\epsilon)} f(x-y)\,\mathrm{d}S^{n-1}(y) \\
&= -⨍_{\partial B(0,\epsilon)} f(x-y)\,\mathrm{d}S(y).
\end{aligned}
$$

We conclude that

$$\lim_{\epsilon\to 0} K_\epsilon = -f(x). \tag{37}$$

We conclude that (36) holds. ☐

From the proof of the above Theorem 10.4, we obtain the following corollary

Corollary 10.5 (Fundamental property of the fundamental solution)**.** *Let $U \subset \mathbb{R}^n$ open and $f \in C^0(U)$. For every $x \in U$,*

$$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} f(y)\nabla_y \Phi(x-y)\cdot \frac{y-x}{|y-x|}\,\mathrm{d}S(y) = -f(x);$$

or, equivalentely,

$$\lim_{\epsilon\to 0}\int_{\partial B(0,\epsilon)} f(x-y)\nabla_y \Phi(y)\cdot \frac{y}{|y|}\,\mathrm{d}S(y) = -f(x).$$

Proof. See (37). ☐

Remark 10.6. (1) We are inverting the Laplacian, in some sense.

(2) If $\phi \in C^2(\mathbb{R}^n)$ is harmonic (e.g., a harmonic polynomial), then $\triangle(u+\phi) = \triangle u = f$. So, the above integral formula (35) selects one solution among many.

(3) Our integral formula is only solving $\triangle u = f$ on the whole space $\mathbb{R}^n$ and only for f with compact support. There are various direction in which to extend this result: to f continuous (in fact, f of class C^2 looks weird: if $u \in C^2$, then we expect $f = \triangle u \in C^0$) [see **§13.34**]; to f not with compact support (in fact, the selected u does not have compact support anyway); to the case of f defined only on an open subset U [see **§10.17**]. I hope we can see all these extensions in this course: we will check!

Exercise 10.7. Using the ideas in the proof of Theorem 10.4, prove the following statement: *If $u \in C^2(\mathbb{R}^n)$ is such that $\triangle u \in C^0_c(\mathbb{R}^n)$, then, for every $x \in \mathbb{R}^n$,*

$$u(x) = -\int_{\mathbb{R}^n} \Phi(x-y)\triangle u(y)\,\mathrm{d}y.$$

Can you weaken the condition "$\triangle u \in C^0_c(\mathbb{R}^n)$"?

Hint. Look also at Theorem 10.39. ◊

§10.7. Mean value property. See Section 6 first.

Theorem 10.8 (Mean value property for harmonic functions)**.** *Let $U \subset \mathbb{R}^n$ be open and $u \in C^2(U)$. The following statements are equivalent:*

(i) u is harmonic, i.e., $\triangle u = 0$;

(ii) $\forall x \in U$, $\forall r > 0$, such that $\bar{B}(x,r) \subset U$,

$$u(x) = \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y).$$

(iii) $\forall x \in U$, $\forall r > 0$, such that $\bar{B}(x,r) \subset U$,

$$u(x) = \fint_{B(x,r)} u(y)\,\mathrm{d}y.$$

Proof. As in (16), we set

$$\phi_u(x;r) := \fint_{B(x,r)} u(z)\,\mathrm{d}z, \quad \text{and} \quad \psi_u(x;r) := \fint_{\partial B(x,r)} u(z)\,\mathrm{d}S(z).$$

Then, from (19) we have,

$$\partial_r \phi_u(x;r) = \frac{n}{r}(\psi_u(x;r) - \phi_u(x;r)), \tag{38}$$

$$\partial_r \psi_u(x;r) = \frac{r}{n}\phi_{\triangle_x u}(x;r). \tag{39}$$

(i) ⇒ *(ii)*: By (39), if $\triangle u = 0$, then $\partial_r \psi_u = 0$, i.e., $r \mapsto \psi_u(x;r)$ is constant and, since $B(x,r) \subset U$, we have

$$u(x) \overset{(17)}{=} \lim_{\epsilon\to 0} \psi_u(x;\epsilon) = \psi_u(x;r).$$

(ii) ⇒ *(iii)*: Let $x \in U$ and $r > 0$ be such that $\bar{B}(x,r) \subset U$. Then

$$\begin{aligned}\fint_{B(x,r)} u(y)\,\mathrm{d}y &= \frac{1}{r^n\omega_n}\int_0^r \int_{\partial B(x,s)} u(y)\,\mathrm{d}S(y)\,\mathrm{d}r \\ &= \frac{n}{r^n}\int_0^r \fint_{\partial B(x,s)} u(y)\,\mathrm{d}S(y) s^{n-1}\,\mathrm{d}r \\ &\overset{(ii)}{=} u(x)\frac{n}{r^n}\int_0^r s^{n-1}\,\mathrm{d}r = u(x).\end{aligned}$$

(iii) ⇒ *(ii)*: Let $x \in U$ and $r > 0$ be such that $\bar{B}(x,r) \subset U$. From (38), we obtain,

$$0 = \partial_r \phi_u(x;r) = \frac{n}{r}(\psi_u(x;r) - u(x)),$$

i.e., $\psi_u(x;r) = u(x)$.

$(ii) \Rightarrow (i)$: Let $x \in U$. From (39), we have

$$\triangle u(x) \overset{(17)}{=} \lim_{r\to 0} \phi_{\triangle u}(x;r) \overset{(39)}{=} \lim_{r\to 0} \frac{n}{r}\partial_r \psi_u(x;r) \overset{(ii)}{=} 0.$$

□

Remark 10.9. If $\partial B(x,r) \subset U$ but $B(x,r) \not\subset U$, then the mean value formula does not need to hold. Find an example.

Exercise 10.10. For all $n \geq 2$, compute

$$\psi_\Phi(0;r) := \fint_{\partial B(0,r)} \Phi(y)\,\mathrm{d}S(y),$$

where Φ is the fundamental solution of the Laplace equation. Is $r \mapsto \psi_\Phi(0;r)$ constant? ♢

Exercise 10.11. In the proof of Theorem 10.8, we have shown $(ii) \Rightarrow (i)$. Give a direct proof of $(iii) \Rightarrow (i)$. ♢

Exercise 10.12. Show the equivalence $(iii) \Leftrightarrow (ii)$ assuming only $u \in C^0(U)$. ♢

§10.8. Strong maximum principle.

Theorem 10.13 (Strong maximum principle - First version)**.** *Suppose $U \subset \mathbb{R}^n$ is open and connected, and $u \in C^2(U;\mathbb{R})$ is harmonic. If there exists $x \in U$ is such that $u(x) = \sup_U u$, then u is constant.*

Proof. Define

$$M = \sup_U u \quad \text{and} \quad W = \{x \in U : u(x) = M\} = u^{-1}(\{M\}).$$

Since u is continuous, then W is closed in U.

We claim that W is open. If $x \in W$, then there is $r > 0$ such that $B(x,r) \subset U$, because U is open. Therefore, since u is harmonic, by Theorem 10.8 we have

$$M = u(x) \overset{10.8.(iii)}{=} \fint_{B(x,r)} u(y)\,\mathrm{d}y \overset{(*)}{\leq} M$$

where the inequality $(*)$ is strict[1] unless $u(y) = M$ for all $y \in B(x,r)$. It follows that $B(x,r) \subset W$.

In conclusion, since U is connected and $W \subset U$ is both open and closed, then $W \in \{\emptyset, U\}$. If W is not empty, then $W = U$, i.e., u is constant. □

Exercise 10.14. Show that, if $u \in C^0(B(0,r);\mathbb{R})$ is such that $u \leq M$ on $B(0,r)$, then $\fint_{B(x,r)} u(y)\,\mathrm{d}y \leq M$, with equality if and only if $u = M$ on $B(0,r)$. ♢

Theorem 10.15 (Strong maximum principle - Second version)**.** *Suppose $U \subset \mathbb{R}^n$ is open and bounded, and $u \in C^2(U;\mathbb{R}) \cap C^0(\bar{U};\mathbb{R})$ is harmonic. Then*

$$\max_{\bar{U}} u = \max_{\partial U} u.$$

Proof. We clearly have $\max_{\bar{U}} u \geq \max_{\partial U} u$. Suppose that there exists $x \in U$ with $u(x) = \max_{\bar{U}} u$. We need to show that there is also a point on ∂U with the same value. Let $C \subset U$ be the connected component of U containing x. Since C is open and connected, by Theorem 10.13, u is constant on C. Therefore, if $y \in \partial C$, then $u(y) = u(x) = \max_{\bar{U}} u$. Since $\mathbb{R}^n$ is locally connected, then $\partial C \subset \partial U$ (see Exercise 10.16). Therefore,

$$\max_{\bar{U}} u = u(x) = \max_{\partial C} u \leq \max_{\partial U} u.$$

□

[1] Indeed, if $|B(x,r) \setminus W| > 0$, then

$$\begin{aligned}\fint_{B(x,r)} u(y)\,\mathrm{d}y &= \frac{1}{|B(x,r)|}\left(\int_{B(x,r)\cap W} u(y)\,\mathrm{d}y + \int_{B(x,r)\setminus W} u(y)\,\mathrm{d}y\right)\\ &< \frac{1}{|B(x,r)|}\left(M|B(x,r)\cap W| + M|B(x,r)\setminus W|\right) = M.\end{aligned}$$

Exercise 10.16. Show the following statement: *Let X be a locally connected topological space and $U \subset X$ open. Let $C \subset U$ be a connected component of U. Then $\partial C \subset \partial U$.*

After this, show that the hypothesis of X being locally connected is necessary. In other words, find an example of a topological space X that is not locally connected, and $U \subset X$ open that has a connected component $C \subset U$ with $\partial C \cap \partial U = \emptyset$. ♢

Solution to Exercise 10.16. Let $x \in \partial C$. Then $x \in \bar{U}$. If $x \in U$, then there exists a connected neighborhood V of x with $V \subset U$. Since $x \in \partial C$, then $V \cap C \neq \emptyset$. We then have $C \cup V \subset U$ is connected. Since C is a maximal connected subset of U, we conclude $V \subset C$, in contradiction with $x \in \partial C$. □

Exercise 10.17. State and prove the *Strong Minimum Principle* for harmonic functions, both in the first and second versions. ♢

Exercise 10.18. Show the strong maximum and minimum principles for harmonic functions without using the mean value property, i.e., explicitly using that $\triangle u = 0$.

Hint: First consider $u \in C^2(U;\mathbb{R})$ with $\triangle u > 0$ in U. Suppose $x \in U$ is such that $u(x) = \max_U u$. Then $t = 0$ is a point of maximum for $t \mapsto u(x + te_j)$, for each $j \in \{1, \dots, n\}$. Therefore $\frac{\partial^2 u}{\partial x_j^2}(x) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} u(x + te_j) \le 0$. ♢

Remark 10.19. In Theorem 10.15, we actually prove more than what we claim. In fact, do not need "$u \in C^2(U;\mathbb{R}) \cap C^0(\bar{U};\mathbb{R})$ harmonic", but only $u \in C^0(\bar{U};\mathbb{R})$ satisfying the mean value property, as stated in Theorem 10.8.(ii). See also Exercise 10.12. Maybe, in your personal notes you can rewrite these statements in the more general form.

§10.9. Subharmonic functions.

Exercise 10.20. Let $U \subset \mathbb{R}^n$ open. A function $u \in C^2(U)$ is *subharmonic* if and only if $\triangle u \ge 0$ on U. A function $u \in C^2(U)$ is *superharmonic* if and only if $\triangle u \le 0$ on U. (Notice the inversion between "sub" and "$\ge$").

State and prove modified versions of Theorem 10.8 and Theorem 10.13 for subharmonic functions.

After this, do the same for superharmonic functions. ♢

§10.10. Monotonicity of Laplace's boundary value problem.

Corollary 10.21. *Let $U \subset \mathbb{R}^n$ open, bounded and connected. Let $u \in C^2(U;\mathbb{R}) \cap C^0(\bar{U};\mathbb{R})$ and $g \in C^0(\partial U;\mathbb{R})$ be such that*

$$\begin{cases} \triangle u = 0 & \text{in } U, \\ u = g & \text{on } U. \end{cases}$$

If g is not constant and $g \ge 0$ on ∂U, then $u > 0$ on U.

Exercise 10.22. Write a proof of Corollary 10.21. ♢

Exercise 10.23. Write and prove a version of Corollar 10.21 for sub- and superharmonic functions. ♢

§10.11. Uniqueness for the Poisson equation.

Theorem 10.24 (Uniqueness for the Poisson equation)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, $f \in C^0(U)$ and $g \in C^0(\partial U)$.*

There exists at most one solution in $C^2(U) \cap C^0(\bar{U})$ to the boundary value problem

$$\begin{cases} -\triangle u = f & \text{in } U, \\ u = g & \text{on } U. \end{cases} \tag{40}$$

Proof. Suppose $u_1, u_2 \in C^2(U) \cap C^0(\bar{U})$ are solutions to (40). Then if u is the real or imaginary part of $u_1 - u_2$, then $u \in C^2(U;\mathbb{R}) \cap C^0(\bar{U};\mathbb{R})$ and u solves

$$\begin{cases} \triangle u = 0 & \text{in } U, \\ u = 0 & \text{on } U. \end{cases}$$

Apply the maximum principle Theorem 10.15 and the minimum principle (or the maximum principle to $-u$), to conclude that $u = 0$ on U. Therefore, $u_1 = u_2$. □

§10.12. Smoothness of harmonic functions.

Theorem 10.25. *Let $U \subset \mathbb{R}^n$ be open and $u \in C^0(U)$. Suppose that u satisfies the spherical mean-value property, i.e.,*

$$\forall x \in U \ \forall r > 0 \ \ s.t.\ \bar{B}(x,r) \subset U: \qquad u(x) = \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y). \tag{41}$$

Then $u \in C^\infty(U)$.

Proof. Let $\{\eta_\epsilon\}_{\epsilon>0}$ be the family of standard mollifiers. Recall that $\eta_\epsilon(x) = \tilde{\eta}(|x|/\epsilon)/\epsilon^n$ for a compactly supported function $\tilde{\eta} \in C^\infty(\mathbb{R})$ with $\tilde{\eta} \geq 0$, $\mathrm{spt}(\tilde{\eta}) \subset [-1,1]$, and $\int_{\mathbb{R}} \tilde{\eta}(t)\,\mathrm{d}t = 1$. In particular, it follows that $\mathrm{spt}(\eta_\epsilon) \subset \bar{B}(0,\epsilon)$ and $\int_{\mathbb{R}^n} \eta_\epsilon(x)\,\mathrm{d}x = 1$.

We will show that the spherical mean-value property (41) implies that, for every $\epsilon > 0$, $u * \eta_\epsilon = u$ on

$$U_\epsilon = \{x \in U : \mathrm{dist}(x, \partial U) > \epsilon\}.$$

Since $u * \eta_\epsilon \in C^\infty(U_\epsilon)$ by Proposition 8.2, we will conclude $u \in C^\infty(U)$.

So, if $x \in U_\epsilon$, then

$$\begin{aligned}
u * \eta_\epsilon(x) &= \int_{B(x,\epsilon)} u(y)\eta_\epsilon(x-y)\,\mathrm{d}y \\
&= \int_0^\epsilon \int_{\partial B(x,r)} u(y)\eta_\epsilon(x-y)\,\mathrm{d}S(y)\,\mathrm{d}r \\
&= \int_0^\epsilon \tilde{\eta}_\epsilon(r) \left(\int_{\partial B(x,r)} \mathrm{d}S(y) \right) \fint_{\partial B(x,r)} u(y)\,\mathrm{d}S(y)\,\mathrm{d}r \\
&\overset{(41)}{=} u(x) \int_0^\epsilon \tilde{\eta}_\epsilon(r) \int_{\partial B(x,r)} \mathrm{d}S(y)\,\mathrm{d}r \\
&= u(x) \int_0^\infty \int_{\partial B(x,r)} \eta_\epsilon(x-y)\,\mathrm{d}S(y)\,\mathrm{d}r \\
&= u(x) \int_{\mathbb{R}^n} \eta_\epsilon(x-y)\,\mathrm{d}y = u(x).
\end{aligned}$$

□

Corollary 10.26. *Harmonic functions are C^∞ smooth.*

§10.13. Analyticity of harmonic functions.

Proposition 10.27 (Estimates on derivatives of harmonic functions)**.** *Let $U \subset \mathbb{R}^n$ open and $u \in C^\infty(U)$ harmonic. For every $x \in U$ and $r > 0$ with $B(x,r) \subset U$, and for every $\alpha \in \mathbb{N}^n$ with $|\alpha| = k$ we have*

$$|\mathrm{D}^\alpha u(x)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B(x,r))}, \qquad \text{where} \quad \begin{aligned} C_0 &= \tfrac{1}{\omega_n}, \text{ and} \\ C_k &= \tfrac{(2^{n+1}nk)^k}{\omega_n} \text{for } k \geq 1. \end{aligned} \tag{42}$$

The proof of Proposition 10.27 will come after a few lemmas.

Lemma 10.28 (Derivative of harmonic functions)**.** *Every partial derivative of a harmonic function is harmonic.*

Proof. Let $U \subset \mathbb{R}^n$ open and $u \in C^\infty(U)$ harmonic. If $j \in \{1, \ldots, n\}$, then

$$\triangle(\partial_j u) = \sum_{k=1}^n \partial_k^2 \partial_j u = \partial_j \sum_{k=1}^n \partial_k^2 u = \partial_j \triangle u = 0,$$

because $\partial_j \partial_k = \partial_k \partial_j$ for all j, k. So, all first-order derivatives of u are harmonic. Iterating, all derivatives of harmonic functions are harmonic harmonic functions. □

Lemma 10.29 (Case $k = 0$)**.** *Let $U \subset \mathbb{R}^n$ open and $u \in C^\infty(U)$ harmonic. For every $x \in U$ and $r > 0$ with $B(x,r) \subset U$,*

$$|u(x)| \leq \frac{1}{\omega_n r^n} \|u\|_{L^1(B(x,r))}. \tag{43}$$

Proof. The ball mean-value property for harmonic functions gives

$$|u(x)| = \left| \fint_{B(x,r)} u(y)\,\mathrm{d}y \right| \le \frac{1}{\omega_n r^n} \|u\|_{L^1(B(x,r))}.$$

□

Lemma 10.30 (Case $k = 1$: Estimate of first derivatives of harmonic functions)**.** *Let $U \subset \mathbb{R}^n$ open and $u \in C^\infty(U)$ harmonic. For every $x \in U$ and $r > 0$ with $B(x,r) \subset U$, and for every $j \in \{1,\dots,n\}$ and $\theta \in (0,1)$, we have*

$$|\partial_j u(x)| \le \frac{1}{\omega_n (\theta r)^n} \int_{\partial B(x,\theta r)} |u(y)|\,\mathrm{d}S(y). \tag{44}$$

As a consequence, we also have

$$|\partial_j u(x)| \le \frac{2^{n+1} n}{\omega_n} \frac{1}{r^{n+1}} \|u\|_{L^1(B(x,r))}. \tag{45}$$

Proof. Fix $j \in \{1,\dots,n\}$. By Lemma 10.28, $\partial_j u$ is also harmonic, and thus it has the mean-value property. Therefore, for every $\theta \in (0,1)$,

$$\begin{aligned}
|\partial_j u(x)| &= \left| \fint_{B(x,\theta r)} \partial_j u(y)\,\mathrm{d}y \right| \\
&= \frac{1}{\omega_n (\theta r)^n} \left| \int_{B(x,\theta r)} \mathrm{div}(u e_j)(y)\,\mathrm{d}y \right| \\
&= \frac{1}{\omega_n (\theta r)^n} \left| \int_{\partial B(x,\theta r)} u(y) e_j \cdot \frac{y-x}{|y-x|}\,\mathrm{d}S(y) \right| \\
&\le \frac{1}{\omega_n (\theta r)^n} \int_{\partial B(x,\theta r)} |u(y)|\,\mathrm{d}S(y).
\end{aligned}$$

This shows (44). Next, we use the fact that, if $y \in \partial B(x,\theta r)$, then $B(y,(1-\theta)r) \subset B(x,r)$. Thus, applying Lemma 10.29 to the running estimates, we get

$$\begin{aligned}
|\partial_j u(x)| &\le \frac{1}{\omega_n (\theta r)^n} \int_{\partial B(x,\theta r)} |u(y)|\,\mathrm{d}S(y) \\
&\overset{(43)}{\le} \frac{1}{\omega_n (\theta r)^n} \int_{\partial B(x,\theta r)} \frac{1}{\omega_n ((1-\theta)r)^n} \|u\|_{L^1(B(y,(1-\theta)r))}\,\mathrm{d}S(y) \\
&\le \frac{1}{\omega_n (\theta r)^n} \frac{1}{\omega_n ((1-\theta)r)^n} n\omega_n (\theta r)^{n-1} \|u\|_{L^1(B(x,r))} \\
&= \frac{n}{\omega_n \theta (1-\theta)^n r^{n+1}} \|u\|_{L^1(B(x,r))}
\end{aligned}$$

All in all, if we take $\theta = 1/2$, we have (45). □

Proof of Proposition 10.27. We shall argue by induction over k.

For $k = 0$ and $k = 1$, we have already Lemma 10.29 and Lemma 10.30, respectively.

Let $m \ge 1$ and assume that (42) holds for all $k \le m$. We now prove (42) for $k = m+1$. Let $\alpha \in \mathbb{N}^n$ with $|\alpha| = m+1$. Then there is $\beta \in \mathbb{N}^n$ with $|\beta| = m$ and $j \in \{1,\dots,n\}$ such that $\mathrm{D}^\alpha = \mathrm{D}^\beta \frac{\partial}{\partial x_j}$.

The function $\mathrm{D}^\beta u$ is harmonic by Lemma 10.28. We apply Lemma 10.30 to the function $\mathrm{D}^\beta u$ and get that, for $\theta \in (0,1)$,

$$\begin{aligned}
|\mathrm{D}^\alpha u(x)| &= |\partial_j \mathrm{D}^\beta u(x)| \\
&\overset{(44)}{\le} \frac{1}{\omega_n (\theta r)^n} \int_{\partial B(x,\theta r)} |\mathrm{D}^\beta u(y)|\,\mathrm{d}S(y) \\
\text{[inductive hypothesis]} &\le \frac{1}{\omega_n (\theta r)^n} \frac{C_m}{((1-\theta)r)^{n+m}} \|u\|_{L^1(B(x,r))} \int_{\partial B(x,\theta r)} \mathrm{d}S(y) \\
&= C_m n \frac{1}{(1-\theta)^{n+m}\theta} \frac{1}{r^{n+m+1}} \|u\|_{L^1(B(x,r))}.
\end{aligned}$$

Taking $\theta = \frac{1}{m+1}$ and $C_m = \frac{(2^n nm)^m}{\omega_n}$, we get

$$
\begin{aligned}
|\mathrm{D}^\alpha u(x)| &\leq \frac{(2^n nm)^m}{\omega_n} \frac{n(m+1)^{m+n+1}}{m^{m+n}} \frac{1}{r^{n+m+1}} \|u\|_{L^1(B(x,r))} \\
[\text{computations}] &\leq \frac{(2^n n(m+1))^{m+1}}{\omega_n} \frac{1}{r^{n+m+1}} \|u\|_{L^1(B(x,r))}.
\end{aligned}
$$

□

Exercise 10.31. Let $a, b < 0$ and define $\phi : (0,1) \to \mathbb{R}$, $\phi(\theta) = \theta^a \theta^b$. Find the minimum and the point of minimum of ϕ on $(0,1)$.

Solution: $\theta_m = \frac{a}{a+b}$ and $\phi(\theta_m) = \left(\frac{a}{a+b}\right)^{a+b}$. ◊

Theorem 10.32 (Analyticity)**.** *Harmonic functions are analytic.*

Proof. NOTA BENE: This proof contains a mistake. Find it and correct it. The correct proof is in Section **§18.1**, page 122.

Let $U \subset \mathbb{R}^n$ be open and $u \in C^\infty(U)$ a harmonic function. Fix $\hat{x} \in U$ and set $\hat{r} = \frac{1}{4}\mathrm{dist}(\hat{x}, \partial U)$. We claim that there exists $\epsilon \in (0,1)$ such that, if

$$r < \epsilon \hat{r},$$

then the Taylor series of u centered at $\hat{x}$ converges on $B(\hat{x}, r)$ to u, that is, for every $x \in B(\hat{x}, r)$,

$$u(x) = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\alpha|=k} \frac{\mathrm{D}^\alpha u(\hat{x})}{\alpha!}(x-\hat{x})^\alpha.$$

To this aim, define the reminder function

$$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{\mathrm{D}^\alpha u(\hat{x})}{\alpha!}(x-\hat{x})^\alpha.$$

For every $x \in B(\hat{x}, r)$ there exists $t_x \in [0,1]$ such that

$$R_N(x) = \sum_{|\alpha|=N} \frac{\mathrm{D}^\alpha u(\hat{x} + t(x-\hat{x}))}{\alpha!}(x-\hat{x})^\alpha.$$

Using Proposition 10.27, we make the following estimate: since $\hat{x} + t(x-\hat{x}) \in B(\hat{x}, r) \subset B(\hat{x}, \hat{r})$,

$$
\begin{aligned}
|R_N(x)| &\leq \sum_{|\alpha|=N} \frac{|\mathrm{D}^\alpha u(\hat{x} + t(x-\hat{x}))|}{\alpha!}|x-\hat{x}|^N \\
&\overset{(42)}{\leq} \frac{(2^{n+1}nN)^N \|u\|_{L^1(B(\hat{x},\hat{r}))}}{\omega_n \hat{r}^{n+N}} r^N \sum_{|\alpha|=N} \frac{1}{\alpha!} \\
&\overset{(46)}{=} \frac{\|u\|_{L^1(B(\hat{x},\hat{r}))}}{\omega_n \hat{r}^n} \epsilon^N (2^{n+1}nN)^N \frac{n^N}{N!} \\
&= \frac{\|u\|_{L^1(B(\hat{x},\hat{r}))}}{\omega_n \hat{r}^n} \frac{\sqrt{2\pi N}(N/e)^N}{N!} \frac{(\epsilon 2^{n+1} n^2 N)^N}{\sqrt{2\pi N}(N/e)^N} \\
&= \frac{\|u\|_{L^1(B(\hat{x},\hat{r}))}}{\omega_n \hat{r}^n} \frac{\sqrt{2\pi N}(N/e)^N}{N!} \frac{(\epsilon 2^{n+1} n^2 e)^N}{\sqrt{2\pi N}},
\end{aligned}
$$

where we have used the Multinomial Theorem

$$n^N = (\sum_{j=1}^{n} 1)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!}. \tag{46}$$

Using Stirling's formula

$$\lim_{N\to\infty} \frac{\sqrt{2\pi N}(N/e)^N}{N!} = 1,$$

we conclude that, if $\epsilon \in (0,1)$ is so that $\epsilon 2^{n+1} n^2 e < 1$, then $\lim_{N\to\infty} |R_N(x)| = 0$. □

§10.14. Liouville's Theorem.

Theorem 10.33 (Liouville's Theorem). *If $u : \mathbb{R}^n \to \mathbb{C}$ is harmonic and bounded, then u is constant.*

Proof. Fix $x \in \mathbb{R}^n$. Then, for all $r > 0$, we have from Proposition 10.27,

$$\begin{aligned} |Du(x)| &\le \sum_{j=1}^n \left| \frac{\partial u}{\partial x_j}(x) \right| \le \frac{nC_1}{r^{n+1}} \int_{B(x,r)} |u(y)| \,\mathrm{d}y \\ &\le \frac{nC_1}{r^{n+1}} \|u\|_{L^\infty(\mathbb{R}^n)} \omega_n r^n = \frac{nC_1 \omega_n \|u\|_{L^\infty(\mathbb{R}^n)}}{r}. \end{aligned}$$

Since this inequality holds for every $r > 0$, then $\mathrm{D}u(x) = 0$. Since x is arbitrary, then $\mathrm{D}u \equiv 0$ on $\mathbb{R}^n$, i.e., u is constant. □

Exercise 10.34. Prove Liouville's Theorem using only the ball mean-value property.
Solution: Fix $\hat{x} \in \mathbb{R}^n$ and $x \in B(\hat{x}, 1)$. Then

$$\begin{aligned} |u(x) - u(\hat{x})| &= \left| \fint_{B(x,r)} u(y)\,\mathrm{d}y - \fint_{B(\hat{x},r)} u(y)\,\mathrm{d}y \right| \\ &= \frac{1}{\omega_n r^n} \left| \int_{B(x,r)\setminus B(\hat{x},r)} u(y)\,\mathrm{d}y - \int_{B(\hat{x},r)\setminus B(x,r)} u(y)\,\mathrm{d}y \right| \\ &\le \frac{1}{\omega_n r^n} \int_{B(\hat{x},r+1)\setminus B(\hat{x},r-1)} |u(y)|\,\mathrm{d}y \\ &\le \|u\|_{L^\infty(\mathbb{R}^n)} \frac{(r+1)^n - (r-1)^n}{r^n} \\ &= \|u\|_{L^\infty(\mathbb{R}^n)} \left((1+1/r)^n - (1-1/r)^n \right) \overset{r\to\infty}{\longrightarrow} 0. \end{aligned}$$

◊

§10.15. Representation formula.

Define

$$C^2_b(\mathbb{R}^n) = \{u \in C^2(\mathbb{R}^n) : \|u\|_{L^\infty} < \infty\}.$$

Theorem 10.35. *Suppose $n \ge 3$ and fix $f \in C^2_c(\mathbb{R}^n)$. The solutions u of $-\triangle u = f$ in $C^2_b(\mathbb{R}^n)$ are exactly all functions in the family*

$$\left\{ x \mapsto \int_{\mathbb{R}^n} \Phi(x-y) f(y)\,\mathrm{d}y + c : c \in \mathbb{C} \right\}$$

Proof. We already know that the function $\tilde{u}(x) = \int_{\mathbb{R}^n} \Phi(x-y)f(y)\,\mathrm{d}y$ is of class C^2 and solves $-\triangle \tilde{u} = f$, see Theorem 10.4.

We claim that $\tilde{u} \in C^2_b(\mathbb{R}^n)$. Indeed, if $\mathrm{spt}(f) \subset \mathbb{R}^n \setminus B(x,R)$, then

$$\begin{aligned} |\tilde{u}(x)| &= \left| \int_{\mathrm{spt}(f)} \Phi(x-y) f(y)\,\mathrm{d}y \right| \\ &\le \|f\|_{L^\infty} \Phi(R) |\mathrm{spt}(f)|. \end{aligned}$$

So, if $\mathrm{spt}(f) \subset B(0,L)$, then for every x with $|x| \ge L$, we have $\tilde{u}(x) \le \|f\|_{L^\infty} |\mathrm{spt}(f)| \Phi(|x| - L)$. Therefore, $\lim_{x\to\infty} |\tilde{u}(x)| = 0$. In particular, $\tilde{u}$ is bounded on $\mathbb{R}^n$.

If $-\triangle u = f$ and if u is bounded, then $u - \tilde{u}$ is bounded and $\triangle(u - \tilde{u}) = 0$. By Liouville's Theorem 10.33, $u - \tilde{u}$ is constant. □

§10.16. Harnack's Inequality.

Theorem 10.36 (Harnack's Inequality). *Let $U \subset \mathbb{R}^n$ be open and $V \Subset U$ an open and connected subset. There exists $C > 0$ (depending on U and V) such that, if $u : U \to [0, +\infty)$ is harmonic on U and non-negative, then*

$$\sup_V u \le C \inf_V u,$$

or, equivalentely,

$$\forall x, y \in V: \qquad u(x) \le Cu(y).$$

Proof. Let $r = \frac{1}{4}\mathrm{dist}(V, \partial U)$. If $x, y \in V$ are such that $|x-y| < r$, then, by the mean value property Theorem 10.8,

$$\begin{aligned} u(x) &= \fint_{B(x,2r)} u(z)\,\mathrm{d}z \\ &= \frac{1}{\omega_n (2r)^n}\left(\int_{B(y,r)} u(z)\,\mathrm{d}z + \int_{B(x,2r)\setminus B(y,r)} u(z)\,\mathrm{d}z\right) \\ [\text{since } u \ge 0] &\ge \frac{1}{2^n}\fint_{B(y,r)} u(z)\,\mathrm{d}z = \frac{1}{2^n} u(y). \end{aligned}$$

Since $\bar{V}$ is compact and connected, there is a finite family of points $x_1, \dots, x_N \in V$ such that $|x_j - x_{j+1}| < r$ and $V \subset \bigcup_{j=1}^N B(x_j, r)$; see Exercise 10.38. Then, whenever $x, y \in V$,

$$u(x) \ge \left(\frac{1}{2^n}\right)^{N+1} u(y).$$

□

Remark 10.37. Here are a few comments to Theorem 10.36:

(1) If $\inf_V u = 0$, then $u = 0$.
(2) If $\inf_V u = 1$, then $\sup_V u \le C$, where the constant C is determined by U and V, but independent on u.
(3) Example on $\mathbb{R}$: take $V = (1,2)$ and $U = (0,4)$ and make a picture of affine maps that must be positive on U.

Exercise 10.38. Let $K \subset \mathbb{R}^n$ be compact and connected and $r > 0$. Show that there is a finite family of points $x_1, \dots, x_N \in K$ such that $|x_j - x_{j+1}| < r$ and $K \subset \bigcup_{j=1}^N B(x_j, r)$. ◊

§10.17. Green's function. (See section 2.2.4 in Evans' book)

Theorem 10.39 (Representation formula using Green's function)**.** *Let $U \subset \mathbb{R}^n$ be open, bounded and with C^1 boundary ∂U. Suppose that there exists a* corrector function $\phi : \bar{U} \times \bar{U} \to \mathbb{R}$, $\phi(x,y) = \phi^x(y)$, *such that, for every $x \in U$,*

$$\begin{cases} \triangle \phi^x = 0 & \text{in } U, \\ \phi^x(y) = \Phi(y - x) & \forall y \in \partial U, \end{cases}$$

where Φ is the fundamental solution in $\mathbb{R}^n$. Define the Green's function of U *as*

$$G : \{(x,y) \in U \times U : x \ne y\} \to \mathbb{R}, \qquad G(x,y) = \Phi(y-x) - \phi^x(y).$$

If $u \in C^2(\bar{U})$, then, for every $x \in U$,

$$u(x) = -\int_{\partial U} u(y) \nabla G(x,y) \cdot \nu_U(y)\,\mathrm{d}S(y) - \int_U \triangle u(y) G(x,y)\,\mathrm{d}y. \tag{47}$$

In particular, if $f \in C^0(U)$, $g \in C^0(\partial U)$ and $u \in C^2(\bar{U})$ are such that

$$\begin{cases} -\triangle u = f & \text{in } U, \\ u = g & \text{on } \partial U, \end{cases}$$

then, for all $x \in U$,

$$u(x) = -\int_{\partial U} g(y) \nabla G(x,y) \cdot \nu_U(y)\,\mathrm{d}S(y) - \int_U f(y) G(x,y)\,\mathrm{d}y.$$

Proof. If $\epsilon > 0$ is such that $B(x,\epsilon) \Subset U$, then

$$\int_U \triangle u(y) G(x,y)\,\mathrm{d}y = \int_{B(x,\epsilon)} \triangle u(y) G(x,y)\,\mathrm{d}y + \int_{U \setminus B(x,\epsilon)} \triangle u(y) G(x,y)\,\mathrm{d}y.$$

where both $\triangle u$ and $G(x,\cdot)$ belong to $L^1(U)$ and thus

$$\lim_{\epsilon\to 0}\int_{B(x,\epsilon)} \triangle u(y)G(x,y)\,\mathrm{d}y = 0.$$

Set $V_\epsilon = U \setminus B(x,\epsilon)$. Then

$$\begin{aligned}
&\int_{V_\epsilon} \triangle u(y)G(x,y)\,\mathrm{d}y = \int_{V_\epsilon} \mathrm{div}_y(\nabla u(y)G(x,y))\,\mathrm{d}y - \int_{V_\epsilon} \nabla u(y)\cdot\nabla_y G(x,y)\,\mathrm{d}y\\
&= \int_{V_\epsilon} \mathrm{div}_y(\nabla u(y)G(x,y))\,\mathrm{d}y - \int_{V_\epsilon} \mathrm{div}_y(u(y)\nabla_y G(x,y))\,\mathrm{d}y\\
&\qquad + \int_{V_\epsilon} u(y)\underbrace{\triangle_y G(x,y)}_{=0}\,\mathrm{d}y\\
&= \int_{\partial V_\epsilon} \nabla u(y)G(x,y)\cdot\nu_{V_\epsilon}(y)\,\mathrm{d}S(y) - \int_{\partial V_\epsilon} u(y)\nabla_y G(x,y)\cdot\nu_{V_\epsilon}(y)\,\mathrm{d}S(y)\\
&= -\int_{\partial B(x,\epsilon)} \nabla u(y)G(x,y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}S(y) + \int_{\partial U}\nabla u(y)\underbrace{G(x,y)}_{=0}\cdot\nu_U(y)\,\mathrm{d}S(y)\\
&\qquad - \int_{\partial U} u(y)\nabla_y G(x,y)\,\mathrm{d}S(y) + \int_{\partial B(x,\epsilon)} u(y)\nabla_y G(x,y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}S(y)\\
&= -\int_{\partial B(x,\epsilon)} \nabla u(y)G(x,y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}S(y) - \int_{\partial U} u(y)\nabla_y G(x,y)\,\mathrm{d}S(y)\\
&\qquad + \int_{\partial B(x,\epsilon)} u(y)\nabla_y\Phi(y-x)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}S(y) + \int_{\partial B(x,\epsilon)} u(y)\nabla_y\phi^x(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}S(y).
\end{aligned}$$

Since we have

$$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} G(x,y)\nabla u(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y = 0, \tag{48}$$

$$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} u(y)\nabla_y\Phi(y-x)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y = -u(x), \qquad \text{(see Cor. 10.5)} \tag{49}$$

$$\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} u(y)\nabla_y\phi^x(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y = 0, \tag{50}$$

then we conclude that

$$\begin{aligned}
\int_U \triangle u(y)G(x,y)\,\mathrm{d}y &= \lim_{\epsilon\to 0}\int_{B(x,\epsilon)} \triangle u(y)G(x,y)\,\mathrm{d}y + \int_{U\setminus B(x,\epsilon)} \triangle u(y)G(x,y)\,\mathrm{d}y\\
&= -u(x) - \int_{\partial U} u(y)\nabla_y G(x,y)\,\mathrm{d}y.
\end{aligned}$$

So, we have (47). □

Remark 10.40. If U is connected, then for each $x\in U$ there exists at most one corrector function ϕ^x, by Theorem 10.24. At the moment, however, we do not know whether corrector function exists.

Exercise 10.41. Show (48), (49), and (50), from the proof of the *Representation formula using Green's function*, Theorem 10.39:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} G(x,y)\nabla u(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y = 0,\\
&\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} u(y)\nabla_y\Phi(y-x)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y = -u(x),\\
&\lim_{\epsilon\to 0}\int_{\partial B(x,\epsilon)} u(y)\nabla_y\phi^x(y)\cdot\frac{y-x}{|y-x|}\,\mathrm{d}y = 0.
\end{aligned}$$

◊

§10.18. Symmetry of Green's functions.

Theorem 10.42 (Symmetry of Green's functions)**.** *Let $U \subset \mathbb{R}^n$ be open and bounded, with C^1 boundary ∂U. Let G be the Green's function on U. Then*

$$\forall x, y \in U \text{ with } x \neq y \qquad G(x,y) = G(y,x).$$

Proof. Fox $x, y \in U$ with $x \neq y$. Define

$$v(z) = G(x,z) = \Phi(z-x) - \phi^x(z) \quad \text{and} \quad w(z) = G(y,z) = \Phi(z-y) - \phi^y(z),$$

where Φ is the fundamental solution on $\mathbb{R}^n$ and ϕ^y is a corrector function. Then $\triangle v = \triangle w = 0$ in $U \setminus \{x,y\}$ and $u = v = 0$ on ∂U.

Let $\epsilon > 0$ be such that $\bar{B}(x,\epsilon) \cup \bar{B}(y,\epsilon) \subset U$ and $\bar{B}(x,\epsilon) \cap \bar{B}(y,\epsilon) = \emptyset$. Set $V_\epsilon = U \setminus (\bar{B}(x,\epsilon) \cup \bar{B}(y,\epsilon))$. Then[2]

$$\begin{aligned} 0 &= \int_{V_\epsilon} (v\triangle w - w\triangle v)\,\mathrm{d}z \\ &\overset{(51)}{=} \int_{\partial V_\epsilon} (v\nabla w - w\nabla v)\cdot \nu_{V_\epsilon}\,\mathrm{d}S(z) \\ &= -\int_{\partial B(x,\epsilon)} (v\nabla w - w\nabla v)\cdot \frac{z-x}{|z-x|}\,\mathrm{d}S(z) + \int_{\partial B(y,\epsilon)} (v\nabla w - w\nabla v)\cdot \frac{z-y}{|z-y|}\,\mathrm{d}S(z). \end{aligned}$$

Since, using also Corollary 10.5,

$$\begin{aligned} &\lim_{\epsilon\to 0} \int_{\partial B(x,\epsilon)} v\nabla w \cdot \frac{z-x}{|z-x|}\,\mathrm{d}S(z) = 0, \\ &\lim_{\epsilon\to 0} \int_{\partial B(x,\epsilon)} w\nabla v \cdot \frac{z-x}{|z-x|}\,\mathrm{d}S(z) = -w(x), \end{aligned} \tag{52}$$

then we obtain $w(x) = v(y)$. □

Exercise 10.43. Show (52) in the proof of the *Symmetry of Green's functions*, Theorem 10.42:

$$\begin{aligned} &\lim_{\epsilon\to 0} \int_{\partial B(x,\epsilon)} v\nabla w \cdot \frac{z-x}{|z-x|}\,\mathrm{d}S(z) = 0, \\ &\lim_{\epsilon\to 0} \int_{\partial B(x,\epsilon)} w\nabla v \cdot \frac{z-x}{|z-x|}\,\mathrm{d}S(z) = -w(x). \end{aligned}$$

◊

§10.19. Uniqueness by Energy methods.

Theorem 10.44 (Uniqueness by Energy methods)**.** *Let $U \subset \mathbb{R}^n$ be open, bounded and with C^1 boundary ∂U. Fix $f \in C^0(U)$ and $g \in C^0(\partial U)$. Then there exists at most one solution u in $C^2(\bar{U})$ to the boundary value problem*

$$\begin{cases} -\triangle u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases} \tag{53}$$

Proof. Suppose $u_1, u_2 \in C^2(\bar{U})$ are solutions to (53) . Then $u = u_1 - u_2 \in C^2(\bar{U})$ is so that $\triangle u = 0$ in U and $u = 0$ on ∂U. Therefore,

$$\begin{aligned} 0 &= \int_U u\triangle u\,\mathrm{d}x \\ &= \int_U \mathrm{div}(u\nabla u)\,\mathrm{d}x - \int_U \nabla u\cdot\nabla u\,\mathrm{d}x \\ &= \int_{\partial U} u\nabla u\cdot\nu_U\,\mathrm{d}S(x) - \int_U |\nabla u|^2\,\mathrm{d}x \end{aligned}$$

[2]

$$\int_U (v\triangle w - w\triangle v)\,\mathrm{d}x = \int_{\partial U} (v\nabla w - w\nabla v)\cdot\nu_U\,\mathrm{d}S(x). \tag{51}$$

$$= -\int_U |\nabla u|^2 \, \mathrm{d}x.$$

Or:

$$\begin{aligned}
\int_U |\nabla u|^2 \, \mathrm{d}x &= \int_U \nabla u \cdot \nabla u \, \mathrm{d}x \\
&= \int_U (\operatorname{div}(u\nabla u) - u\triangle u) \, \mathrm{d}x \\
&= \int_{\partial U} u\nabla u \, \mathrm{d}S(x) = 0.
\end{aligned}$$

Hence, $\nabla u = 0$ in U, that is, since U is connected, u is constant. Since $u = 0$ on ∂U, then $u = 0$ in U.

□

Remark 10.45. What is the difference between Theorem 10.44 and Theorem 10.24?

§10.20. Dirichlet's principle.

Theorem 10.46 (Dirichlet's principle)**.** *Let $U \subset \mathbb{R}^n$ be open, bounded and with C^1 boundary ∂U. Fix $f \in C^0(U)$ and $g \in C^0(\partial U)$.*

Define the Dirichlet's energy functional

$$\mathtt{E}_f : C^2(\bar{U}) \to \mathbb{R}, \qquad \mathtt{E}_f(u) = \int_U (\frac{1}{2}|\nabla u|^2 - uf) \, \mathrm{d}x.$$

Define the admissible set

$$\mathscr{A}_g = \{u \in C^2(\bar{U}) : u = g \text{ on } \partial U\}.$$

For all $u \in C^2(\bar{U})$, the following statements are equivalent:

(i) (53) *holds, that is,*

$$\begin{cases} -\triangle u = f & \text{in } U, \\ u = g & \text{on } \partial U. \end{cases}$$

(ii) $u \in \mathscr{A}_g$ and $\mathtt{E}_f(u) \le \mathtt{E}_f(w)$ for all $w \in \mathscr{A}_g$, i.e., u is the (unique) arg-min of $\mathtt{E}_f$ on $\mathscr{A}_g$.

Proof. *(i)* ⇒ *(ii)*: The fact that $u \in \mathscr{A}_g$ is clear. We need to show that $\mathtt{E}_f(u) \le \mathtt{E}_f(w)$ for all $w \in \mathscr{A}_g$. If $w \in \mathscr{A}_g$, then

$$\begin{aligned}
\mathtt{E}_f(u) - \mathtt{E}_f(w) &= \int_U \big((|\nabla u|^2/2 - uf) - (|\nabla w|^2/2 - wf)\big) \, \mathrm{d}x \\
&= \frac{1}{2}\int_U (|\nabla u|^2 - |\nabla w|^2) \, \mathrm{d}x - \int_U (u - w) f \, \mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
-\int_U (u-w) f \, \mathrm{d}x &= \int_U (u-w)\triangle u \, \mathrm{d}x \\
&= \int_U (\operatorname{div}((u-w)\nabla u) - \nabla(u-w)\cdot\nabla u) \, \mathrm{d}x \\
&= \underbrace{\int_{\partial U} (u-w)\nabla u \cdot \nu_U \, \mathrm{d}S}_{=0 \text{ because } uw, \in \mathscr{A}_g} - \int_U |\nabla u|^2 \, \mathrm{d}x + \int_U \nabla w \cdot \nabla u \, \mathrm{d}x.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathtt{E}_f(u) - \mathtt{E}_f(w) &= -\int_U (|\nabla u|^2/2 + |\nabla w|^2/2 - \nabla w \cdot \nabla u) \, \mathrm{d}x \\
&= -\int_U |\nabla u - \nabla w|^2 \, \mathrm{d}x \le 0.
\end{aligned}$$

Remark 10.47. We see from this calculation that, if $\mathtt{E}_f(u) = \mathtt{E}_f(w)$, then $u = w$. This is indeed the proof of the uniqueness in disguised terms.

(ii) $\Rightarrow$ *(i)*: Fix $\phi \in C_c^\infty(U)$ and define $\iota_\phi : \mathbb{R} \to \mathbb{R}$ by

$$\iota_\phi(\tau) := \mathtt{E}_f(u + \tau\phi).$$

Notice that $u + \tau\phi \in \mathscr{A}_g$ for all $\tau \in \mathbb{R}$, because $\phi = 0$ on ∂U. If ι_ϕ is differentiable at 0, then $\frac{\mathrm{d}}{\mathrm{d}\tau}\iota_\phi(0) = 0$. In fact, we have

$$\begin{aligned}\iota_\phi(\tau) &= \int_U (|\nabla(u+\tau\phi)|^2/2 - (u+\tau\phi)f)\,\mathrm{d}x \\ &= \tau^2 \int_U |\nabla\phi|^2/2\,\mathrm{d}x + \tau \int_U (\nabla u \cdot \nabla\phi - \phi f)\,\mathrm{d}x + \int_U (|\nabla u|^2/2 - uf)\,\mathrm{d}x.\end{aligned}$$

So, $\iota_\phi(\tau)$ is polynomial in τ and it has a minimum at 0 if and only if $\int_U (\nabla u \cdot \nabla\phi - \phi f)\,\mathrm{d}x = 0$, where

$$\begin{aligned}\int_U (\nabla u \cdot \nabla\phi - \phi f)\,\mathrm{d}x &= \underbrace{\int_{\partial U} \phi\nabla u \cdot \nu_U\,\mathrm{d}S}_{=0 \text{ because } \phi = 0 \text{ on } \partial U} - \int_U \phi(f + \triangle u)\,\mathrm{d}x \\ &= -\int_U \phi \cdot (f + \triangle u)\,\mathrm{d}x.\end{aligned}$$

We conclude that, if u is as in (ii), then

$$\forall\phi \in C_c^\infty(U), \qquad \int_U \phi \cdot (f + \triangle u)\,\mathrm{d}x = 0. \tag{54}$$

By the Fundamental Theorem of Calculus of Variations 3.6, we (54) is equivalent to $-\triangle u = f$ on U. □

Remark 10.48. We have not proven yet that the boundary value problem (53) has any solution at all. Do you have an idea of how to prove it? Think about it.

11. Heat Equation

For $U \subset \mathbb{R}^n$ open and $I \subset \mathbb{R}$ open interval, e.g., $I = (0, +\infty)$, $u \in C^2(U \times I)$ and $f \in C^0(U \times I)$, the *heat equation* is

$$\partial_t u - \triangle u = 0 \text{ in } U \times I,$$

and the *nonhomogeneous heat equation* is

$$\partial_t u - \triangle u = f \text{ in } U \times I.$$

In these expressions, $\partial_t u$ is the derivative of u in the direction of I in the product $U \times I$, while the Laplacian $\triangle u = \triangle_x u$ is with respect to the space variable, that is, in the directions U in $U \times I$. In other words, if (x, t) are the coordinates of $U \times I$, with $x \in U$ and $t \in I$, then

$$(\partial_t - \triangle)u = \partial_t u - \triangle u = \frac{\partial u}{\partial t}(x, t) - \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2}(x, t).$$

We call $\partial_t - \triangle$ the *heat operator*. The heat operator $\partial_t - \triangle$ is a parabolic linear differential operator of order 2.

§11.1. Example of solutions.

Exercise 11.1. Define

$$\psi_+ : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}, \qquad \psi_+(x, t) = \frac{1}{t^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

Show that $(\partial_t - \triangle)\psi_+ = 0$ in $\mathbb{R}^n \times (0, +\infty)$. Draw a graph of $x \mapsto \psi_+(x, t)$ for positive t when $n = 1$ (We will see that this function represents a forward propagation: this is why we have a plus.) $\diamondsuit$

Exercise 11.2. Define

$$\psi_- : \mathbb{R}^n \times (0, +\infty) \to \mathbb{R}, \qquad \psi_-(x, t) = \frac{1}{t^{n/2}} \exp\left(\frac{|x|^2}{4t}\right).$$

Show that $(\partial_t - \triangle)\psi_+ = 0$ in $\mathbb{R}^n \times (0, +\infty)$. Draw a graph of $x \mapsto \psi_+(x, t)$ for positive t when $n = 1$. (We will see that this function represents a backward propagation: this is why we have a minus.) $\diamondsuit$

§11.2. Symmetries of the heat operator. Let $U \subset \mathbb{R}^n$ open, $I \subset \mathbb{R}$ an open interval, $u \in C^2(U \times I)$, $O \in \mathtt{O}(n)$ and $b \in \mathbb{R}^n$, $\tau \in \mathbb{R}$, $\lambda \in \mathbb{R} \setminus \{0\}$. Define $\tilde{u}(y, s) = u(\lambda O y + b, \lambda^2 s + \tau)$. Then $\tilde{u} \in C^2(O^{-1}(U - b) \times (I - \tau))$ and

$$(\partial_s - \triangle_y)\tilde{u}(y, s) = \lambda^2(\partial_t u - \triangle_x u)(\lambda O y + b, \lambda^2 s + \tau). \tag{55}$$

Exercise 11.3. Show the identity (55). $\diamondsuit$

Exercise 11.4. Let $U \subset \mathbb{R}^n$ open, $I \subset \mathbb{R}$ an open interval, $u \in C^2(U \times I)$, $O \in \mathtt{O}(n)$ and $b \in \mathbb{R}^n$, $\tau \in \mathbb{R}$ and $\lambda, \sigma \in \mathbb{R} \setminus \{0\}$ Define

$$\tilde{u}(y, s) = u(\lambda O y + b, \sigma s + \tau).$$

Compute $(\partial_t - \triangle)\tilde{u}$ in terms of $(\partial_t - \triangle)u$. Determine for which choices of transformations we have that, if u is a solution to the heat equation, i.e., $(\partial_t - \triangle)u = 0$, then $\tilde{u}$ is also a solution to the heat equation, i.e., $(\partial_t - \triangle)\tilde{u} = 0$. $\diamondsuit$

Exercise 11.5. Show that, if $\lambda > 0$ and if $(\partial_t - \triangle)u = 0$, then $(\partial_t - \triangle)\tilde{u} = 0$, where $\tilde{u}(x, t) = u(\lambda x, \lambda^2 t)$. $\diamondsuit$

§11.3. Fundamental solution for the heat equation. The *fundamental solution for the heat equation* is the function $\Phi : \mathbb{R}^n \times \mathbb{R} \setminus \{(0, 0)\} \to [0, +\infty)$ defined by

$$\Phi(x, t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ 0 & \text{otherwise, i.e., } (x, t) \in (\mathbb{R}^n \times (-\infty, 0]) \setminus \{(0, 0)\}. \end{cases} \tag{56}$$

Exercise 11.6. Find the formula for the fundamental solution for the heat equation by yourself in the following way: look for Φ of the form $\Phi(x,t) = \frac{1}{t^\alpha} v\left(\frac{|x|}{t^\beta}\right)$, or $\Phi(x,t) = \frac{1}{t^\alpha} v\left(\frac{|x|^2}{t^\beta}\right)$, with $(\partial_t - \triangle_x)\Phi = 0$ in $\mathbb{R}^n \times (0,+\infty)$. ♢

Lemma 11.7.

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}. \tag{57}$$

Proof. See stackexchange. □

Exercise 11.8. Find a proof (by yourself or in the literature) for (57), that is,

$$\int_{\mathbb{R}} e^{-x^2}\,\mathrm{d}x = \sqrt{\pi}.$$

♢

Proposition 11.9 (Properties of the fundamental solution)**.** *The function Φ defined in* (56) *has the following properties:*

(1) $\Phi \in C^\infty(\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\})$.

(2) for every $t > 0$ and $x \in \mathbb{R}^n$,

$$\nabla_x \Phi(x,t) = -\frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)\frac{x}{2t} = -\Phi(x,t)\frac{x}{2t},$$

$$\partial_t \Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)\left(\frac{|x|^2}{4t^2} - \frac{n}{2}\frac{1}{4\pi t}\right) = \Phi(x,t)\left(\frac{|x|^2}{4t^2} - \frac{n}{2t}\right),$$

$$\mathrm{D}_x^2 \Phi(x,t) = \Phi(x,t)\left(\frac{x \otimes x}{4t^2} - \frac{\mathrm{Id}}{2t}\right).$$

(3) $(\partial_t - \triangle)\Phi = 0$ *in* $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$.

(4) for every $t > 0$, $\int_{\mathbb{R}^n} \Phi(x,t)\,\mathrm{d}x = 1$

Proof. Proof of 1: To show that Φ is smooth, we proceed as follows. Define

$$\mathscr{F} = \left\{ \phi : \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to \mathbb{R} : \begin{array}{l} \exists P(x,t) \text{ polynomial with} \\ \phi(x,t) = P(x, t^{-1/2}) \exp\left(-\frac{|x|^2}{4t}\right) \text{ for } t > 0, \\ \text{while } \phi(x,t) = 0 \text{ for } t \le 0 \end{array} \right\}$$

Then we have the following two facts (whose proof is left as an exercise): First, $\mathscr{F} \subset C^0(\mathbb{R}^n \times \mathbb{R})$. Second, if $\phi \in \mathscr{F}$, then $\frac{\partial \phi}{\partial x_j}, \frac{\partial \phi}{\partial t} \in \mathscr{F}$, for all $j \in \{1,\dots,n\}$. We conclude that $\mathscr{F} \subset C^\infty(\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\})$. Since $\Phi \in \mathscr{F}$, we the proof is complete.

Proofs of 2 and 3 are left as exercise.

Proof of 4: Fix $t > 0$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^n} \Phi(x,t)\,\mathrm{d}x &= \int_{\mathbb{R}^n} \exp\left(-\frac{|x|^2}{4t}\right)\frac{\mathrm{d}x}{(4\pi t)^{n/2}} \\ [y = \frac{x}{2\sqrt{t}}] &= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} \exp(-|y|^2)\,\mathrm{d}y \\ &= \frac{1}{\pi^{n/2}} \prod_{j=1}^n \int_{\mathbb{R}} \exp(-y_j^2)\,\mathrm{d}y_j \\ &\overset{(57)}{=} 1. \end{aligned}$$

□

Exercise 11.10. Complete the proof of part 1 in Proposition 11.9. Specifically, define

$$\mathscr{F} = \left\{ \phi : \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to \mathbb{R} : \begin{array}{l} \exists P(x,t) \text{ polynomial with} \\ \phi(x,t) = P(x, t^{-1/2}) \exp\left(-\frac{|x|^2}{4t}\right) \text{ for } t > 0, \\ \text{while } \phi(x,t) = 0 \text{ for } t \le 0 \end{array} \right\}.$$

Then show

(1) $\mathscr{F} \subset C^0(\mathbb{R}^n \times \mathbb{R})$.
(2) if $\phi \in \mathscr{F}$, then $\frac{\partial \phi}{\partial x_j}, \frac{\partial \phi}{\partial t} \in \mathscr{F}$, for all $j \in \{1, \dots, n\}$.

Conclude that $\mathscr{F} \subset C^\infty(\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\})$. ♢

Exercise 11.11. Prove parts 2 and 3 in Proposition 11.9. Specifically, show that for every $t > 0$ and $x \in \mathbb{R}^n$,

$$\begin{aligned} \nabla_x \Phi(x,t) &= -\frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) \frac{x}{2t} = -\Phi(x,t)\frac{x}{2t}, \\ \partial_t \Phi(x,t) &= \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right)\left(\frac{|x|^2}{4t^2} - \frac{n}{2}\frac{1}{4\pi t}\right) = \Phi(x,t)\left(\frac{|x|^2}{4t^2} - \frac{n}{2t}\right), \\ \mathrm{D}^2_x \Phi(x,t) &= \Phi(x,t)\left(\frac{x \otimes x}{4t^2} - \frac{\mathrm{Id}}{2t}\right), \end{aligned}$$

where Φ is the fundamental solution of the heat operator. Conclude that $(\partial_t - \triangle_x)\Phi = 0$ in $\mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. ♢

§11.4. Solution to the Cauchy problem.

Theorem 11.12. *Let $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and define $u : \mathbb{R}^n \times (0, +\infty) \to \mathbb{C}$ as*

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,\mathrm{d}y = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4t}\right) g(y)\,\mathrm{d}y,$$

for $x \in \mathbb{R}^n$ and $t > 0$, where Φ is the fundamental solution for the heat operator (56).

(1) $u \in C^\infty(\mathbb{R}^n \times (0,+\infty))$ and, for every $\alpha \in \mathbb{N}^{n+1}$,

(58) $$\mathrm{D}^\alpha u(x,t) = \int_{\mathbb{R}^n} \mathrm{D}^\alpha_{x,t} \Phi(x-y,t) g(y)\,\mathrm{d}y.$$

(2) $(\partial_t - \triangle)u = 0$ in $\mathbb{R}^n \times (0,+\infty)$;
(3) for each $\hat{x} \in \mathbb{R}^n$,

(59) $$\lim_{\substack{(x,t)\to(\hat{x},0)\\ x \in \mathbb{R}^n, t>0}} u(x,t) = g(x).$$

In particular, u has a continuous extension $u \in C^0(\mathbb{R}^n \times [0,+\infty)) \cap C^\infty(\mathbb{R}^n \times (0,+\infty))$ and

$$\begin{cases} \partial_t u - \triangle u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Proof. Proof of 1:

$$K(x,t;y) = \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right) g(y).$$

For $R > 0$ and $\epsilon > 0$, we have, for every $x \in \mathbb{R}^n$ with $|x| < R$ and all $t > \epsilon$,

$$|K(x,t;y)| \le h_{R,\epsilon}(y) := \begin{cases} \frac{\|g\|_{L^\infty}}{(4\pi\epsilon)^{n/2}} & \text{if } |y| \le R \\ \frac{\|g\|_{L^\infty}}{(4\pi\epsilon)^{n/2}} \exp\left(-\frac{(|y|-R)^2}{4\epsilon}\right) & \text{if } |y| > R \end{cases}$$

Since $h_R \in L^1(\mathbb{R}^n)$, satisfy the conditions of Theorem **(3.3.2)**, so $u \in C^0(B(0,R) \times (\epsilon, +\infty))$. Since this holds for every $R > 0$ and $\epsilon > 0$, we conclude $u \in C^0(\mathbb{R}^n \times (0,+\infty))$. To prove the smoothness of u, one uses the same strategy as in the proof of Proposition 11.9.1; see Exercise 11.13. Also (58) follows from Proposition 11.9.

Proof of 2: From (58), we have

$$(\partial_t - \triangle)u(x,t) = \int_{\mathbb{R}^n} (\partial_t - \triangle_x)\Phi(x-y,t) g(y)\,\mathrm{d}y = 0,$$

because $(\partial_t - \triangle_x)\Phi(x-y,t) = 0$ for every $y \in \mathbb{R}^n$ and $t > 0$, thanks to Proposition 11.9.3.

Proof of 3: Fix $\hat{x} \in \mathbb{R}^n$ and $\epsilon > 0$. Since g is continuous, there exists $\delta > 0$ such that $|g(y) - g(\hat{x})| < \epsilon$ for all $y \in B(\hat{x}, \delta)$.

$$|u(x,t) - g(\hat{x})| = \left| \int_{\mathbb{R}^n} \Phi(x-y,t) g(y)\,\mathrm{d}y - g(\hat{x}) \int_{\mathbb{R}^n} \Phi(x-y,t)\,\mathrm{d}y \right|$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}^n} \Phi(x-y,t)|g(y)-g(\hat{x})| \,\mathrm{d}y \\
&= \int_{B(\hat{x},\delta)} \Phi(x-y,t) \underbrace{|g(y)-g(\hat{x})|}_{<\epsilon} \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n \setminus B(\hat{x},\delta)} \Phi(x-y,t)|g(y)-g(\hat{x})| \,\mathrm{d}y \\
&\leq \epsilon \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t) \,\mathrm{d}y}_{=1} + \underbrace{\int_{\mathbb{R}^n \setminus B(\hat{x},\delta)} \Phi(x-y,t)|g(y)-g(\hat{x})| \,\mathrm{d}y}_{=:J_\delta(x)} .
\end{aligned}$$

If $x \in B(\hat{x}, \delta/2)$, then, for all $y \in \mathbb{R}^n \setminus B(\hat{x}, \delta)$,

$$|y-x| \geq |y-\hat{x}| - |\hat{x}-x| \geq |y-\hat{x}| - \delta/2 \geq |y-\hat{x}| - |y-\hat{x}|/2 = |y-\hat{x}|/2. \tag{60}$$

Hence, if $x \in B(\hat{x}, \delta)$,

$$\begin{aligned}
J_\delta(x) &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n \setminus B(\hat{x},\delta)} \frac{\exp(-\frac{|x-y|^2}{4t})}{(4\pi t)^{n/2}} \,\mathrm{d}y \\
&\overset{(60)}{\leq} \frac{2\|g\|_{L^\infty}}{(4\pi)^{n/2}} \int_{\mathbb{R}^n \setminus B(\hat{x},\delta)} \frac{\exp(-\frac{|y-\hat{x}|^2}{16t})}{t^{n/2}} \,\mathrm{d}y \\
[z = \frac{y-\hat{x}}{\sqrt{t}}] &\leq \frac{2\|g\|_{L^\infty}}{(4\pi)^{n/2}} \underbrace{\int_{\mathbb{R}^n \setminus B(\hat{x},\delta/\sqrt{t})} \exp(-\frac{|z|^2}{16}) \,\mathrm{d}y}_{=:E(\delta,t)} .
\end{aligned}$$

Since $\delta > 0$ and since the integrand in $E(\delta, t)$ is integrable over $\mathbb{R}^n$, then $\lim_{t\to 0^+} E(\delta, t) = 0$. Therefore, there exists $\tau > 0$ such that $E(\delta, t) < \epsilon$ for all $t \in (0, \tau)$. All in all, we conclude that

$$\forall \epsilon > 0 \exists \delta, \tau > 0 \forall x \in B(\hat{x}, \delta/2) \ \forall t \in (0,\tau) \qquad |u(x,t) - g(\hat{x})| \leq \epsilon + \frac{2\|g\|_{L^\infty}}{(4\pi)^{n/2}} \epsilon. \tag{61}$$

(61) is (59). □

Exercise 11.13. Show part 1 in Theorem 11.12. ◊

§11.5. Approximation of the identity. From the proof of Theorem 11.12, we can extract the following lemma:

Lemma 11.14. *Let $T > 0$ and $g \in C^0(\mathbb{R}^n \times [0,T]) \cap L^\infty(\mathbb{R}^n \times [0,T])$. Then, for every $\hat{x} \in \mathbb{R}^n$,*

$$\lim_{\substack{(x,t)\to(\hat{x},0)\\ t>0}} \int_{\mathbb{R}^n} \Phi(z,t) g(x-z,t) \,\mathrm{d}z = g(\hat{x}, 0). \tag{62}$$

Exercise 11.15. Prove Lemma 11.14. ◊

Proof. Define

$$u(x,t) = \int_{\mathbb{R}^n} \Phi(z,t) g(x-z,t) \,\mathrm{d}z = \int_{\mathbb{R}^n} \Phi(x-z,t) g(z,t) \,\mathrm{d}z.$$

Fix $\hat{x} \in \mathbb{R}^n$ and $\epsilon > 0$. Since g is continuous, there exists $\delta > 0$ such that $|g(y,t) - g(\hat{x},0)| < \epsilon$ for all $y \in B(\hat{x}, \delta)$ and all $t \in [0, \delta]$. Then

$$\begin{aligned}
|u(x,t) - g(\hat{x},0)| &= \left| \int_{\mathbb{R}^n} \Phi(x-y,t) g(y,t) \,\mathrm{d}y - g(\hat{x},0) \int_{\mathbb{R}^n} \Phi(x-y,t) \,\mathrm{d}y \right| \\
&\leq \int_{\mathbb{R}^n} \Phi(x-y,t) |g(y,t) - g(\hat{x},0)| \,\mathrm{d}y \\
&= \int_{B(\hat{x},\delta)} \Phi(x-y,t) \underbrace{|g(y,t) - g(\hat{x},0)|}_{<\epsilon} \,\mathrm{d}y \\
&\quad + \int_{\mathbb{R}^n \setminus B(\hat{x},\delta)} \Phi(x-y,t) |g(y,t) - g(\hat{x},0)| \,\mathrm{d}y
\end{aligned}$$

$$\leq \epsilon \underbrace{\int_{\mathbb{R}^n} \Phi(x-y,t)\,\mathrm{d}y}_{=1} + \underbrace{\int_{\mathbb{R}^n\setminus B(\hat{x},\delta)} \Phi(x-y,t)|g(y,t)-g(\hat{x},0)|\,\mathrm{d}y}_{=:J_\delta(x)}.$$

If $x \in B(\hat{x},\delta/2)$, then, for all $y \in \mathbb{R}^n \setminus B(\hat{x},\delta)$,

$$|y-x| \geq |y-\hat{x}| - |\hat{x}-x| \geq |y-\hat{x}| - \delta/2 \geq |y-\hat{x}| - |y-\hat{x}|/2 = |y-\hat{x}|/2. \tag{63}$$

Hence, if $x \in B(\hat{x},\delta/2)$,

$$\begin{aligned} J_\delta(x) &\leq 2\|g\|_{L^\infty} \int_{\mathbb{R}^n\setminus B(\hat{x},\delta)} \frac{\exp(-\frac{|x-y|^2}{4t})}{(4\pi t)^{n/2}}\,\mathrm{d}y \\ &\overset{(63)}{\leq} \frac{2\|g\|_{L^\infty}}{(4\pi)^{n/2}} \int_{\mathbb{R}^n\setminus B(\hat{x},\delta)} \frac{\exp(-\frac{|y-\hat{x}|^2}{16t})}{t^{n/2}}\,\mathrm{d}y \\ [z = \frac{y-\hat{x}}{\sqrt{t}}] &= \frac{2\|g\|_{L^\infty}}{(4\pi)^{n/2}} \underbrace{\int_{\mathbb{R}^n\setminus B(\hat{x},\delta/\sqrt{t})} \exp(-\frac{|z|^2}{16})\,\mathrm{d}y}_{=:E(\delta,t)}. \end{aligned}$$

Since $\delta > 0$ and since the integrand in $E(\delta,t)$ is integrable over $\mathbb{R}^n$, then $\lim_{t\to 0^+} E(\delta,t) = 0$. Therefore, there exists $\tau \in (0,\delta)$ such that $E(\delta,t) < \epsilon$ for all $t \in (0,\tau)$. All in all, we conclude that

$$\forall\epsilon > 0 \exists\delta,\tau > 0 \forall x \in B(\hat{x},\delta/2)\ \forall t \in (0,\tau) \qquad |u(x,t) - g(\hat{x},0)| \leq \epsilon + \frac{2\|g\|_{L^\infty}}{(4\pi)^{n/2}}\epsilon. \tag{64}$$

(64) is (62). □

§11.6. Solution to the nonhomogeneous Cauchy problem. A solution for the nonhomogeneous Cauchy problem (66) is constructed as follows. For every $s > 0$ let u_s be the solution constructed in the previous Theorem for the Cauchy problem

$$\begin{cases} \partial_t u_s - \triangle u_s = 0 & \text{in } \mathbb{R}^n \times (s,+\infty), \\ u_s = f(\cdot,s) & \text{on } \mathbb{R}^n \times \{s\}. \end{cases}$$

So, $u_s(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,\mathrm{d}y$. Then, we set $u(x,t) = \int_0^t u_s(x,t)\,\mathrm{d}s$, that is, (65). This strategy is called *Duhamel's principle*.

Theorem 11.16 (Solution to the nonhomogeneous Cauchy problem). *Let $f \in C^{2;1}_c(\mathbb{R}^n \times [0,+\infty))$ and define $u : \mathbb{R}^n \times (0,+\infty) \to \mathbb{R}$ by*

$$\begin{aligned} u(x,t) &= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(y,s)\,\mathrm{d}y\,\mathrm{d}s. \end{aligned} \tag{65}$$

Then the following holds:

(1) $u \in C^{2;1}(\mathbb{R}^n \times (0,+\infty))$;
(2) $(\partial_t - \triangle)u = f$ in $\mathbb{R}^n \times (0,+\infty)$;
(3) for every $\hat{x} \in \mathbb{R}^n$,

$$\lim_{\substack{(x,t)\to(\hat{x},0) \\ x\in\mathbb{R}^n, t>0}} u(x,t) = 0.$$

In particular, u has a continuous extension $u \in C^0(\mathbb{R}^n \times [0,+\infty)) \cap C^\infty(\mathbb{R}^n \times (0,+\infty))$ and

$$\begin{cases} \partial_t u - \triangle u = f & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{66}$$

Proof. Proof of 1. First, notice that the integrand in the definition (65) of u is integrable and thus u is well defined. Moreover,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Phi(z,r) f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r.$$

Define

$$K(x,t;z,r) := \Phi(z,r)f(x-z,t-r)\mathbb{1}_{[0,t]}(r).$$

Then $|K(x,t;z,r)| \leq \|f\|_{L^\infty}\Phi(z,r)$ for all $(x,t),(z,r) \in \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\}$. So, we can apply Theorem 3.3.

Proof of 2. With the support of Theorem 3.3, we can compute, for $x \in \mathbb{R}^n$, $t > 0$, and $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq 2$,

$$\partial_t u(x,t) = \int_{\mathbb{R}^n} \Phi(z,t)f(x-z,0)\,\mathrm{d}z + \int_0^t \int_{\mathbb{R}^n} \Phi(z,r)\partial_t f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r,$$

$$\mathrm{D}^\alpha u(x,y) = \int_0^t \int_{\mathbb{R}^n} \Phi(z,r)\mathrm{D}^\alpha f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r.$$

Therefore, for $x \in \mathbb{R}^n$, $t > 0$ and $\epsilon \in (0,t)$,

$$\begin{aligned}(\partial_t - \triangle)u(x,t) &= \int_{\mathbb{R}^n} \Phi(z,t)f(x-z,0)\,\mathrm{d}z \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \Phi(z,r)(\partial_t - \triangle)f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r \\ &= \underbrace{\int_{\mathbb{R}^n} \Phi(z,t)f(x-z,0)\,\mathrm{d}z}_{:=K} \\ &\quad + \underbrace{\int_0^\epsilon \int_{\mathbb{R}^n} \Phi(z,r)(\partial_t - \triangle)f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r}_{:=J_\epsilon} \\ &\quad + \underbrace{\int_\epsilon^t \int_{\mathbb{R}^n} \Phi(z,r)(\partial_t - \triangle)f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r}_{:=I_\epsilon}.\end{aligned}$$

Now,

$$\begin{aligned}|J_\epsilon| &\leq (\|\partial_t f\|_{L^\infty} + \|\mathrm{D}^2 f\|_{L^\infty}) \int_0^\epsilon \int_{\mathbb{R}^n} \Phi(z,r)\,\mathrm{d}z\,\mathrm{d}r \\ &= \epsilon(\|\partial_t f\|_{L^\infty} + \|\mathrm{D}^2 f\|_{L^\infty}).\end{aligned}$$

Next,

$$\begin{aligned}I_\epsilon &= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(z,r)(\partial_t - \triangle_x)f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r \\ &= \int_\epsilon^t \int_{\mathbb{R}^n} \Phi(z,r)(-\partial_r - \triangle_z)f(x-z,t-r)\,\mathrm{d}z\,\mathrm{d}r \\ &= \int_\epsilon^t \int_{\mathbb{R}^n} (-\partial_r(\Phi(z,r)f(x-z,t-r)) + \partial_r\Phi(z,r)f(x-z,t-r))\,\mathrm{d}z\,\mathrm{d}r \\ &\quad - \int_\epsilon^t \int_{\mathbb{R}^n} \mathrm{div}_z \Big(\Phi(z,r)\nabla_z f(x-z,t-r) - \nabla_z\Phi(z,r)f(x-z,t-r)\Big)\,\mathrm{d}z\,\mathrm{d}r \\ &\quad - \int_\epsilon^t \int_{\mathbb{R}^n} \triangle\Phi(z,r)f(x-z,t-r))\,\mathrm{d}z\,\mathrm{d}r \\ &\overset{(*)}{=} -K + \int_{\mathbb{R}^n} \Phi(z,\epsilon)f(x-z,t-\epsilon)\,\mathrm{d}z\,\mathrm{d}r,\end{aligned}$$

where we used in $(*)$ the fact that Φ solves the homogeneous heat equation and that f has compact support. So, we conclude that

$$\begin{aligned}(\partial_t - \triangle)u(x,t) &= \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \Phi(z,\epsilon)f(x-z,t-\epsilon)\,\mathrm{d}z\,\mathrm{d}r \\ [\text{Lemma 11.14}] &= f(x,t).\end{aligned}$$

Proof of 3. We easily conclude with the following estimate:

$$|u(x,t)| \leq \|f\|_{L^\infty} \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\,\mathrm{d}y\,\mathrm{d}s$$

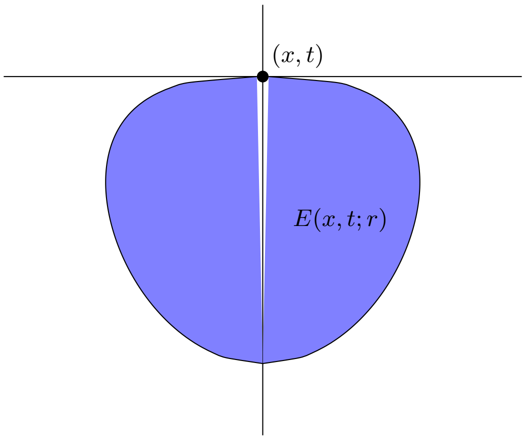


FIGURE 2. The shape of the set $E(x,t;r)$ defined in **§11.7**.

$$[\text{by } 11.9.4] = \|f\|_{L^\infty} t.$$

□

§11.7. Mean-value formula. For $(x,t) \in R^n \times \mathbb{R}$ and $r > 0$, define

$$\begin{aligned} E(x,t;r) &= \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s \leq t,\ \Phi(x-y,t-s) \geq \frac{1}{r^n}\} \\ &= (x,t) + E(0,0;r) = (x,t) + \delta_r(E(0,0;1)), \end{aligned}$$

where $\delta_r(y,s) = (ry, r^2 s)$. Notice that $E(0,0;1) = \{(y,s) : \Phi(-y,-s) \geq 1\}$. See Figure **§11.7** for a drawing of the shape of $E(x,t;r)$. Since

$$\begin{aligned} \Phi(-y,-s) \geq 1 &\Leftrightarrow \frac{(4\pi|s|)^{n/2}}{\exp}\left(-\frac{|y|^2}{4|s|}\right) \geq 1 \\ &\Leftrightarrow \frac{n}{2}\log(4\pi|s|) \leq -\frac{|y|^2}{4|s|} \\ &\Leftrightarrow -\frac{1}{4\pi} < s < 0 \qquad |y|^2 \leq -2n|s|\log(4\pi|s|). \end{aligned}$$

Theorem 11.17. *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be open and $u \in C^{2;1}(\Omega)$ such that $(\partial_t - \triangle)u = 0$ in Ω. Then, for all $(x,t) \in \Omega$ and $r > 0$ such that $E(x,t;r) \subset \Omega$, we have*

$$u(x,t) = \frac{1}{4r^n}\iint_{E(x,t;r)} u(y,s)\frac{|x-y|^2}{(t-s)^2}\,\mathrm{d}y\,\mathrm{d}s.$$

Proof. See [5, pag.53]. □

§11.8. Strong maximum and minimum principles. For $U \subset \mathbb{R}^n$ and $T > 0$, define:

(1) the *closed parabolic cylinder* $\bar{U}_T := \bar{U} \times [0,T]$;
(2) the *parabolic interior* $U_T := U \times (0,T]$ (notice that T is included);
(3) the *parabolic boundary* $\Gamma_T := \bar{U}_T \setminus U_T = (U \times \{0\}) \cup (\partial U \times [0,T))$.

Theorem 11.18 (Strong maximum principle). *Let $U \subset \mathbb{R}^n$ be open and bounded. Let $u \in C^{2;1}(U_T;\mathbb{R}) \cap C^0(\bar{U}_T;\mathbb{R})$ be a real-valued function such that $(\partial_t - \triangle)u = 0$ in U_T. Then*

$$\max_{\bar{U}_T} u = \max_{\Gamma_T} u. \tag{67}$$

Proof. (From Folland [7, Thm 4.16]). In (67), the inequality $\max_{\bar{U}_T} u \geq \max_{\Gamma_T} u$ is clear. We need to show $\max_{\bar{U}_T} u \leq \max_{\Gamma_T} u$.

Let $\epsilon > 0$ and define $v_\epsilon(x,t) = u(x,t) + \epsilon|x|^2$. Let $T' \in (0,T)$. We claim that

$$\max_{\bar{U}_{T'}} v_\epsilon = \max_{\Gamma_{T'}} v_\epsilon. \tag{68}$$

First of all, notice that, in U_T,

$$\partial_t v_\epsilon - \triangle v_\epsilon = \partial_t u - \triangle u - \epsilon \triangle(|x|^2) = -2\epsilon n > 0. \tag{69}$$

Suppose that $(x,t) \in U_{T'}$ is a point of maximum for v_ϵ on $\bar{U}_{T'}$. Then $\triangle v_\epsilon(x,t) \le 0$ and $\partial_t v_\epsilon \ge 0$. However, this is in contradiction with (69). We conclude that (68) must hold.

To prove the statement about u, we see that

$$\begin{aligned} \max_{\bar{U}_{T'}} u &\le \max_{\bar{U}_{T'}} v_\epsilon \\ &\overset{(69)}{=} \max_{\Gamma_{T'}} v_\epsilon \\ &\le \max_{\Gamma_{T'}} u + \epsilon \max_{\bar{U}_{T'}} |x|^2. \end{aligned}$$

Since U is bounded, then $\max_{\bar{U}_{T'}} |x|^2 < \infty$. So, letting $\epsilon \to 0$, we obtain the desired inequality. □

Exercise 11.19. Show the following statement: *Let $U \subset \mathbb{R}^n$ be open, bounded and connected, $T > 0$, and $u \in C^{2;1}(U_T;\mathbb{R}) \cap C^0(\bar{U}_T;\mathbb{R})$ such that $(\partial_t - \triangle)u = 0$ in U_T. If there exists $(x,t) \in U_T$ such that $u(x,t) = \max_{\bar{U}_T} u$, then u is constant on $\bar{U}_t$.*

[This exercise turned out to be more difficult than I expected: It is proven by Evans using the mean value formula from Theorem 11.17.] ♢

Exercise 11.20. Prove the *strong minimum principle* for the heat operator. ♢

Exercise 11.21. Prove the strong maximum principle for the heat operator using the mean-value property; see [5]. ♢

§11.9. Uniqueness on bounded domains.

Theorem 11.22 (Uniqueness on bounded domains)**.** *Suppose $U \subset \mathbb{R}^n$ is open and bounded, and $T > 0$. Let $g \in C^0(\Gamma_T)$ and $f \in C(U_T)$. Then there exist not two distinct solutions in $C^{2;1}(U_T) \cap C(\bar{U}_T)$ to the boundary value problem*

$$\begin{cases} (\partial_t - \triangle)u = f & \text{in } U_T, \\ u = g & \text{on } \Gamma_T. \end{cases} \tag{70}$$

Proof. Let $u_1, u_2 \in C^{2;1}(U_T) \cap C(\bar{U}_T)$ be two solutions to (70). Then $u = u_1 - u_2 \in C^{2;1}(U_T) \cap C(\bar{U}_T)$ satisfies $(\partial_t - \triangle)u = 0$ in U_T and $u = 0$ on Γ_T. By the strong maximum and minimum principles, see **§11.8**, both the real and the imaginary parts of u are zero on U_T, i.e., $u_1 = u_2$. □

§11.10. Maximum and minimum principles for the unbounded Cauchy problem.

Theorem 11.23 (Maximum principle for the unbounded U)**.** *Let $T > 0$. Let $u \in C^{2;1}(\mathbb{R}^n \times (0,T];\mathbb{R}) \cap C^0(\mathbb{R}^n \times [0,T];\mathbb{R})$ be a real-valued function such that $(\partial_t - \triangle)u = 0$ in $\mathbb{R}^n \times (0,T]$. Suppose also that there are $A, a > 0$ such that*

$$u(x,t) \le A \exp(a|x|^2). \tag{71}$$

Then

$$\max_{\mathbb{R}^n \times [0,T]} u = \max_{\mathbb{R}^n} u(\cdot, 0).$$

Proof. Let $0 < b \le T$ be such that

$$\frac{1}{4b} - a > 0, \text{ i.e., } 0 < b < \frac{1}{4a}.$$

We will show that

$$\forall s,t \in [0,T] \text{ with } 0 < t - s < b, \ \sup_{x \in \mathbb{R}^n} u(x,t) \le \sup_{x \in \mathbb{R}^n} u(x,s). \tag{72}$$

Since both (71) and the heat equation are invariant under time translations (see **§11.2**), we can prove (72) for $s = 0$ without loss of generality.

Set $g \in C^0(\mathbb{R}^n)$ by $g(x) = u(x,0)$. Fix $\hat{y} \in \mathbb{R}^n$ and $\hat{t} \in (0,b)$: we need to show that

$$u(\hat{y},\hat{t}) \le \|g\|_{L^\infty(\mathbb{R}^n)}. \tag{73}$$

For $\mu > 0$, define, for $x \in \mathbb{R}^n$ and $t \in (0,b)$,

$$v(x,t) = u(x,t) - \frac{\mu}{(b-t)^{n/2}} \exp\left(\frac{|x-\hat{y}|^2}{4(b-t)}\right).$$

So, we have $v \in C^{2;1}(\mathbb{R}^n \times (0,b);\mathbb{R}) \cap C^0(\mathbb{R}^n \times [0,b);\mathbb{R})$ and $(\partial_t - \triangle)v = 0$ in $\mathbb{R}^n \times (0,b)$; see **§11.1**.

We claim that there exists $r > 0$ such that

$$\sup_{B(\hat{y},r)\times[0,b)} v \le \|g\|_{L^\infty}. \tag{74}$$

By the strong maximum principle, Theorem 11.18, for every $r > 0$, we have

$$\sup_{B(\hat{y},r)\times[0,b)} v = \sup\left\{ v(x,t) : \begin{array}{l} x \in B(\hat{y},r) \text{ and } t = 0, \text{ or} \\ x \in \partial B(\hat{y},r) \text{ and } t \in [0,b) \end{array} \right\}.$$

If $|x-\hat{y}| \le r$, then $v(x,0) = u(x,0) - \frac{\mu}{b^{n/2}} \exp\left(\frac{|x-\hat{y}|^2}{4b}\right) \le g(x) \le \|g\|_{L^\infty}$. If $|x - \hat{y}| = r$ and $t \in [0,b)$, then

$$\begin{aligned} v(x,t) &= u(x,t) - \frac{\mu}{(b-t)^{n/2}} \exp\left(\frac{r^2}{4(b-t)}\right) \\ &\le A\exp(a|x|^2) - \frac{\mu}{b^{n/2}} \exp\left(\frac{r^2}{4b}\right) \\ &\le A\exp(a(|\hat{y}|+r)^2) - \frac{\mu}{b^{n/2}} \exp\left(\frac{r^2}{4b}\right) \\ &= \exp(a(|\hat{y}|+r)^2)\left(A - \frac{\mu}{b^{n/2}} \exp\left(\frac{r^2}{4b} - a(|\hat{y}|+r)^2\right)\right) =: J_r. \end{aligned}$$

Since $\lim_{r\to\infty} J_r = -\infty$, we can choose r large enough that the $J_r < \|g\|_{L^\infty}$. We conclude that, for r large enough, (74) holds.

In particular, from (74) we obtain that, for every $\mu > 0$,

$$u(\hat{y},\hat{t}) - \frac{\mu}{(b-\hat{t})^{n/2}} = v(\hat{y},\hat{t}) \le \|g\|_{L^\infty}. \tag{75}$$

Taking the limit $\mu \to 0$ in (75), we conclude (73). □

Exercise 11.24. State and prove the strong minimum principle for the unbounded Cauchy problem. ◊

§11.11. Uniqueness for the unbounded Cauchy problem.

Theorem 11.25 (Uniqueness for the unbounded Cauchy problem)**.** *Let $T > 0$, $g \in C^0(\mathbb{R}^n)$ and $f \in C^0(\mathbb{R}^n \times [0,T])$. Set*

$$\mathscr{A} = \left\{ u \in C^{2;1}(\mathbb{R}^n \times (0,T]) \cap C^0(\mathbb{R}^n \times [0,T]) : \begin{array}{c} \exists A, a > 0 \text{ s.t.} \\ |u(x,t)| \le A\exp(a|x|^2) \end{array} \right\}. \tag{76}$$

There does not exist two solutions in $\mathscr{A}$ of

$$\begin{cases} (\partial_t - \triangle)u = f & \text{in } \mathbb{R}^n \times (0,T], \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{77}$$

Proof. If $u_1, u_2 \in \mathscr{A}$ solve (77), then $u = u_1 - u_2 \in \mathscr{A}$ is such that $(\partial_t - \triangle)u = 0$ in $\mathbb{R}^n \times (0,T]$ and $u = 0$ for $t = 0$. By the maximum and minimum principles **§11.10**, $u \equiv 0$. □

Remark 11.26. The growth condition (71) is necessary: there are counter-examples. See [10, chapter 7].

Remark 11.27. Notice that in the definition of $\mathscr{A}$ in (76), we require the growth condition (71) both as an upper bound and as a lower bound. You should have noticed this already in Exercise 11.24.

§11.12. Smoothness. The following blue part contains mistakes, so it is dropped from the content of the course. I keep it here for future memory: with a bit of work, we should be able to fix it. Notice that we have other methods to prove that solutions to the heat equation are smooth...

TODO: Fix proof of Lemma 11.28 .

Lemma 11.28. *For $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, define*

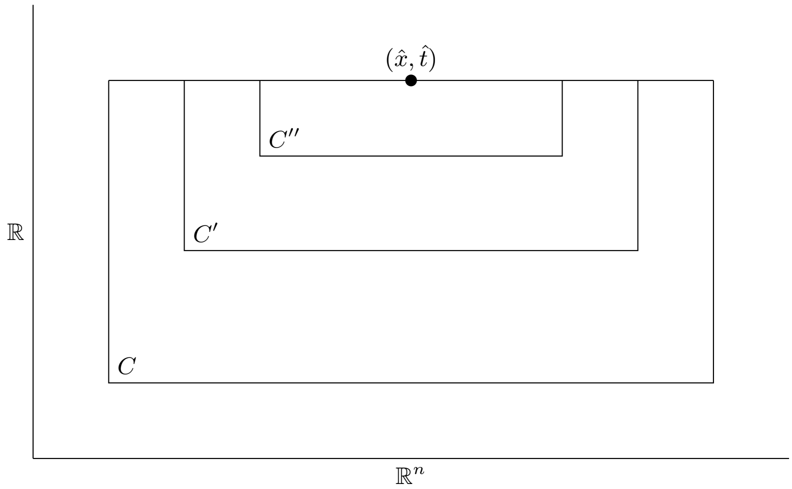
$$C(x,t;r) = \bar{B}(x,r) \times [t - r^2, t] \subset \mathbb{R}^n \times \mathbb{R}.$$

Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ open. Fix $(\hat{x},\hat{t}) \in \Omega$ and $\hat{r} > 0$ such that $C := C(\hat{x},\hat{t};\hat{r}) \subset \Omega$. Define $C' := C(\hat{x},\hat{t};\frac{3}{4}\hat{r})$ and $C'' := C(\hat{x},\hat{t};\frac{1}{2}\hat{r})$.

Let $\zeta \in C_c^\infty(\mathbb{R}^n \times (-\infty,\hat{t}])$ be such that $C' \subset \{\zeta = 1\}$ and $\mathrm{spt}(\zeta) \subset C$.

Then, for every $u \in C^{2;1}(\Omega)$ and all $(x,t) \in C''$,

$$u(x,t) = \int_0^t \int_{\mathbb{R}^n} \Big(\Phi(x-y,t-s)(\partial_s \zeta(y,s) + \triangle_y \zeta(y,s)) + 2\mathrm{D}_y \Phi(x-y,t-s) \mathrm{D}_y \zeta(y,s) \Big) \cdot u(y,s) \,\mathrm{d}y \,\mathrm{d}s$$
$$+ \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\zeta(y,s) \Big(\partial_s u(y,s) - \triangle_y u(y,s) \Big) \,\mathrm{d}y \,\mathrm{d}s.$$



Proof. Assume $\hat{t} > \hat{r} > 0$. Define $v := \zeta \cdot u \in C_c^{2;1}(\mathbb{R}^n \times (-\infty,\hat{t}])$ and compute

$$\tilde{f} := (\partial_t - \triangle)v = (\partial_t - \triangle)\zeta \cdot u + \zeta(\partial_t - \triangle)u - 2\mathrm{D}\zeta \cdot \mathrm{D}u.$$

Set then

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\tilde{f}(y,s) \,\mathrm{d}y \,\mathrm{d}s.$$

By the definition of $\tilde{f}$ and by (65), both v and $\tilde{v}$ are solutions to the Cauchy problem $(\partial_t - \triangle)v = \tilde{f}$ in $\mathbb{R}^n \times (0,\hat{t})$, with $v = 0$ on $\mathbb{R}^n \times \{0\}$. Moreover, we have both $\|v\|_{L^\infty} < \infty$ and $\|\tilde{f}\|_{L^\infty} < \infty$. Therefore, $|\tilde{v}(x,t)| \leq t\|\tilde{f}\|_{L^\infty}$. From Theorem 11.25, it follows that $v = \tilde{v}$.

Since $(x,t) \in C''$, then $\zeta(x,t) = 1$ and

$$u(x,t) = \zeta(x,t)u(x,t) = v(x,t) = \tilde{v}(x,t)$$
$$= \int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\Big((\partial_s - \triangle_y)\zeta(y,s) \cdot u(y,s) + \zeta(y,s) \cdot (\partial_s - \triangle_y)u(y,s)$$
$$- 2\mathrm{D}\zeta(y,s) \cdot \mathrm{D}u(y,s) \Big) \,\mathrm{d}y \,\mathrm{d}s.$$

Moreover,

$$\int_0^t \int_{\mathbb{R}^n} \Phi(x-y,t-s)\mathrm{D}\zeta \cdot \mathrm{D}u \,\mathrm{d}y \,\mathrm{d}s = \int_0^t \int_{\bar{B}(\hat{x},\hat{r})} \Big(\mathrm{div}_y(\Phi(x-y,t-s)\mathrm{D}\zeta(y,s)u(y,s))$$

$$- \mathrm{D}_y\Phi(x-y,t-s)\mathrm{D}\zeta(y,s)u(y,s) \\ - \Phi(x-y,t-s)u(y,s)\triangle\zeta(y,s)\Big)\,\mathrm{d}y\,\mathrm{d}s,$$

where, for all $s < t$,

$$\int_{\bar{B}(\hat{x},\hat{r})} \mathrm{div}_y(\Phi(x-y,t-s)\mathrm{D}\zeta(y,s)u(y,s)\,\mathrm{d}y = 0.$$

□

Theorem 11.29 (Smoothness of solutions to the homogeneous heat equation)**.** *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be open and $u \in C^{2;1}(\Omega)$ such that $(\partial_t - \triangle)u = 0$ in Ω. Then $u \in C^\infty(\Omega)$.*

Proof. By the Lemma 11.28, we have for $(x,t) \in C''$

$$u(x,t) = \iint_C K(x,t;y,s)u(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

where, for $(x,t) \in C''$, $K(x,t;\cdot) \in C^\infty(\bar{C})$. TRUE? Therefore, u is smooth on C'' by Theorem 3.3, or Proposition 3.5. -> C'' is not open! □

Theorem 11.30 (Quantitative smoothness)**.** *Let $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ be an open set. For every $\alpha \in \mathbb{N}^n$, $\ell \in \mathbb{N}$, there exists $C_{\alpha,\ell} \in \mathbb{R}$ such that, for every $u \in C^\infty(\Omega)$ with $(\partial_t - \triangle)u = 0$ in Ω, if $C(x,t;r) \subset \Omega$, then*

$$\sup_{C(x,t;r/2)} \left| \frac{\partial^{|\alpha|}}{\partial x^\alpha} \frac{\partial^\ell}{\partial t^\ell} u \right| \le \frac{C_{\alpha,\ell}}{r^{|\alpha|+2\ell+n+2}} \|u\|_{L^1(C(x,t;r))}. \tag{78}$$

Proof. From Lemma 11.28 we get that, if $u \in C^\infty(C(0,0;1))$, then, for every $(x,t) \in C(0,0;1/2)$,

$$u(x,t) = \iint_{C(0,0;1)} K(x,t;y,s)u(y,s)\,\mathrm{d}y\,\mathrm{d}s,$$

where $K \in C^\infty(\overline{C(0,0;1/2) \times C(0,0;1)})$. By Proposition 3.5, or Theorem 3.3, for every $(x,t) \in C(0,0;1/2)$ we have

$$\mathrm{D}^{\alpha,\ell}_{x,t} u(x,t) = \iint_{C(0,0;1)} \mathrm{D}^{\alpha,\ell}_{x,t} K(x,t;y,s)u(y,s)\,\mathrm{d}y\,\mathrm{d}s.$$

Hence, if we take as constants

$$C_{\alpha,\ell} = \sup\{|\mathrm{D}^{\alpha,\ell}_{x,t}K(x,t;y,s)| : (x,t) \in C(0,0;1/2),\ (y,s) \in C(0,0;1)\},$$

we obtain (78) in this specific case.

In the general case, if $(\hat{x},\hat{t}) \in \Omega$ and $\hat{r} > 0$ are such that $C(\hat{x},\hat{t};\hat{r}) \subset \Omega$, and if $u \in C^\infty(\Omega)$ is such that $(\partial_t - \triangle)u = 0$ in Ω, then the function

$$\hat{u}(x,t) := u(\hat{x} + \hat{r}x, \hat{t} + \hat{r}^2 t)$$

is such that $(\partial_t - \triangle)u = 0$ in $C(0,0;1)$. Therefore, for every $(x,t) \in C(0,0;1/2)$,

$$\begin{aligned} |\mathrm{D}^{\alpha,\ell}_{x,t} u(\hat{x} + \hat{r}x, \hat{t} + \hat{r}^2 t)|\hat{r}^{|\alpha|+2\ell} &= |\mathrm{D}^{\alpha,\ell}_{x,t}\hat{u}(x,t)| \\ &\le C_{\alpha,\ell} \iint_{C(0,0;1)} \hat{u}(y,s)\,\mathrm{d}y\,\mathrm{d}s \\ &= C_{\alpha,\ell} \iint_{C(0,0;1)} u(\hat{x} + \hat{r}y, \hat{t} + \hat{r}^2 s)\,\mathrm{d}y\,\mathrm{d}s \\ &= C_{\alpha,\ell} \iint_{C(\hat{x},\hat{t};\hat{r})} u(\bar{y},\bar{s}) \frac{\mathrm{d}\bar{y}}{\hat{r}^n} \frac{\mathrm{d}\bar{s}}{\hat{r}^2}. \end{aligned}$$

Hence, we conclude with (78). □

Remark 11.31. Notice that this smoothness result tells us that time cannot be inverted!

§11.13. Energy Methods.

Lemma 11.32. *Let $U \subset \mathbb{R}^n$ open and bounded with C^1 boundary ∂U, and let $T > 0$. For $u : U_T \to \mathbb{C}$ define $e_u : [0,T] \to \mathbb{R}$,*

$$e_u(t) = \int_U |u(x,t)|^2 \,\mathrm{d}x,$$

whenever it is well defined. If $u \in C^{2;1}(\bar{U}_T)$ is such that $(\partial_t - \triangle)u = 0$ in U_T, then $e_u \in C^1([0,T])$ and

$$\dot{e}_u(t) = 2\int_{\partial U} u\mathrm{D}u \cdot \nu_U \,\mathrm{d}S - 2\int_U |\mathrm{D}u|^2 \,\mathrm{d}x.$$

If $u = 0$ on $\partial U \times [0,T]$, then $\dot{e}_u(t) \leq 0$ and so, e_t is decreasing.

Proof. Since U is bounded and u is continuous on $\bar{U}$, then $\|u\|_{L^\infty(\bar{U}_T)} < \infty$. Since U is bounded, then constants belong to $L^2(U)$. Therefore, by Theorem 3.3, $e_u \in C^1((0,T))$ and

$$\begin{aligned}\dot{e}_u(t) &= 2\int_U \partial_t u(x,t) u(x,t) \,\mathrm{d}x \\ &= 2\int_U \triangle u(x,t) u(x,t) \,\mathrm{d}x \\ &= 2\int_{\partial U} \mathrm{D}u(x,t) \cdot \nu_U(x) \,\mathrm{d}S(x) - 2\int_U |\mathrm{D}u(x,t)|^2 \,\mathrm{d}x.\end{aligned}$$

□

Theorem 11.33 (Uniqueness by Energy methods)**.** *Let $T > 0$ and $U \subset \mathbb{R}^n$ be an open, bounded subset with C^1 boundary ∂U. Let $f \in C^0(U_T)$ and $g \in C^0(\Gamma_T)$. Then there do not exist two distinct solutions in $C^{2;1}(\bar{U}_T)$ to*

$$\begin{cases} (\partial_t - \triangle)u = f & \text{in } U_T, \\ u = g & \text{on } \Gamma_T. \end{cases} \tag{79}$$

Proof. Suppose there are two solutions $u_1, u_2 \in C^{2;1}(\bar{U}_T)$ to (79). Then $w = u_1 - u_2 \in C^{2;1}(\bar{U}_T)$ solves (79) with $f = 0$ and $g = 0$. By Lemma 11.32, $0 = e_w(0) \geq e_w(t) = \int_U |w(x,t)|^2 \,\mathrm{d}x \geq 0$ for all $t > 0$. It follows that $w = 0$ and thus $u_1 = u_2$. □

§11.14. Backward uniqueness.

Theorem 11.34 (Backward uniqueness)**.** *Let $T > 0$ and $U \subset \mathbb{R}^n$ be an open, bounded subset with C^1 boundary ∂U. Let $g \in C^0(\partial U \times [0,T])$. Suppose that $u_1, u_2 \in C^2(\bar{U}_T)$ solve the Cauchy problem*

$$\begin{cases} (\partial_t - \triangle)u = 0 & \text{in } U_T, \\ u = g & \text{on } \partial U \times [0,T]. \end{cases} \tag{80}$$

If $u_1(x,T) = u_2(x,T)$ for all $x \in U$, then $u_1 = u_2$ in U_T.

Proof. Take $w = u_1 - u_2 \in C^2(\bar{U}_T)$, which solves (80) with $g = 0$. By the Lemma 11.32,

$$\begin{aligned} e_w(t) &= \int_U |w(x,t)|^2 \,\mathrm{d}x; \\ \dot{e}_w(t) &= 2\int_U |\mathrm{D}w(x,t)|^2 \,\mathrm{d}x; \\ \ddot{e}_w(t) &= [...] = 4\int_U |\triangle w(x,t)|^2 \,\mathrm{d}x. \end{aligned}$$

Moreover, using the Hölder inequality,

$$\begin{aligned}\int_U |\mathrm{D}w(x,t)|^2 \,\mathrm{d}x &= \int_U \mathrm{D}w(x,t) \cdot \mathrm{D}w(x,t) \,\mathrm{d}x \\ &= -\int_U \triangle w(x,t) \cdot w(x,t) \,\mathrm{d}x\end{aligned}$$

$$\leq \left(\int_U |\triangle w(x,t)|^2 \, \mathrm{d}x \right)^{1/2} \cdot \left(\int_U |w(x,t)| \, \mathrm{d}x \right)^{1/2}.$$

Hence,

$$\dot{e}_w(t)^2 \leq \ddot{e}_w(t) \cdot e_w(t).$$

Suppose that e_w is not zero on $[0,T]$. Let $(a,b) \subset [0,T]$ be a maximal interval where $e_w > 0$. Then $e_w(a) = 0$. By Lemma 11.35, for all $[t_1,t_2] \subset (a,b)$,

$$e_w(1/2t_1 + 1/2t_2) \leq e_w(t_1)^{1/2} e_w(t_2)^{1/2}.$$

Taking the limit to $t_1 \to a$ and $t_2 \to b$ we obtain $e_w(\frac{a+b}{2}) = 0$, in contradiction with $e_w > 0$ on (a,b). We conclude that $e_w = 0$ on $[0,T]$. □

Lemma 11.35. *Let $I \subset \mathbb{R}$ be an interval and $e : I \to (0,+\infty)$ be a C^2 function with $\dot{e} \leq \mathrm{d}e\dot{e}$. Then, for every $t_1 < t_2$ belonging to I, and every $\tau \in (0,1)$,*

$$e((1-\tau)t_1 + \tau t_2) \leq e(t_1)^{1-\tau} e(t_2)^{\tau}. \tag{81}$$

Proof. Set $f(t) = \log(e(t))$, which is well defined because $e(t) > 0$. Then $f'(t) = \frac{\dot{e}(t)}{e(t)}$ and

$$f''(t) = -\frac{1}{e(t)^2}\dot{e}(t)^2 + \frac{\ddot{e}(t)}{e(t)} \geq \frac{-\ddot{e}e + \ddot{e}e}{e^2} = 0.$$

Therefore f is convex. From the convexity of f, we get (81). □

12. Wave Equation

§12.1. The wave operator. Let $U \subset \mathbb{R}^n$ open and $I \subset \mathbb{R}$ an open interval. For $u : U \times I \to \mathbb{C}$, we call

- the (homogeneous) wave equation $\frac{\partial^2 u}{\partial t^2} - \triangle u = 0$ in $U \times I$;
- the *nonhomogeneous wave equation* $\frac{\partial^2 u}{\partial t^2} - \triangle u = f$ in $U \times I$, for some $f : U \times I \to \mathbb{C}$.

A common abbreviation is to denote the *wave operator* by

$$\Box w = \partial_t^2 - \triangle.$$

§12.2. Symmetries of $\Box$. The wave operator $\Box$ is a linear homogeneous hyperbolic differential operator of order 2.

Define $v(x,t) = u(Ax + a, Bt + b)$ with $A \in \mathtt{GL}(n)$, $a \in \mathbb{R}^n$, $B, b \in \mathbb{R}$ with $AA^T = \lambda^2 \mathrm{Id}$. Then

$$\begin{aligned} \Box v &= (B^2 \partial_t^2 u - \lambda^2 \triangle u)(Ax + a, Bt + b) \\ [\text{If } B^2 = \lambda^2\text{:}] &= \lambda^2 \Box u(Ax + a, Bt + b). \end{aligned}$$

Exercise 12.1. Compute $\Box u$ for $u(x,t) = \exp((a + bi) \cdot x + \phi t)$, where $a, b \in \mathbb{R}^n$ and $\phi \in \mathbb{R}$. ◊

Exercise 12.2. Show that $\Box$ is invariant under the Lorentz group of transformations. Recall that the Lorentz group is the group of linear automorphisms of the Minkowski space. More explicitely, if M is the $(n+1) \times (n+1)$ matrix

$$M = \begin{pmatrix} \mathrm{Id} & 0 \\ 0 & -1 \end{pmatrix},$$

then the Minkowski space is $(\mathbb{R}^n \times \mathbb{R}, M)$ and the Lorentz group is made of matrices $A \in \mathtt{GL}(\mathbb{R}^n \times \mathbb{R})$ such that $A^T M A = M$.

Hint. If it seems too hard, try to solve the exercise at leas for $n = 1$. ◊

§12.3. Examples. For $a, b \in \mathbb{C}$, consider the following functions $u_{a,b} : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$,

$$u_{a,b}(x,t) = \exp(ax + bt),$$

Exercise 12.3. For which $a, b \in \mathbb{C}$ we have $\Box u_{a,b} = 0$?

Hint: $a = b$ or $a = -b$. ◊

Exercise 12.4. For which $a, b \in \mathbb{C}$ we have $u_{a,b}$ 1-periodic (i.e., $u_{a,b}(0) = u_{a,b}(1)$) and $\Box u_{a,b} = 0$?

Hint: $a, b \in 2\pi i \mathbb{Z}$ with $a = b$ or $a = -b$. ◊

Exercise 12.5. For every $k \in \mathbb{N}$ and $\ell > 0$, find $u : [0, \ell] \times \mathbb{R} \to \mathbb{R}$ such that

1. $\Box u = 0$,
2. $u(0,t) = u(\ell, t) = 0$ for all t, and
3. there are $0 \le x_0 < x_1 < \cdots < x_k \le \ell$ such that $u(x_j, t) = 0$ for all j and all t.

These functions are the Harmonics of the string pinched at the two ends. ◊

Exercise 12.6. Let $c \in \mathbb{C}$. Find $u : \mathbb{R}^n \times \mathbb{R} \to \mathbb{C}$ such that

$$\Box u = cu.$$

◊

§12.4. Finite propagation speed.

Theorem 12.7 (Finite propagation speed)**.** *Let $U \subset \mathbb{R}^n$ be open, $\hat{x} \in U$ and $R > 0$. Define*

$$K(\hat{x}, R) = \{(x,t) \in \mathbb{R}^n \times [0, R] : |x| \le R\}.$$

Suppose $\bar{B}(\hat{x}, R) \subset U$ and that $u \in C^\infty(U \times [0, R])$ is such that $\Box u = 0$ in $U \times (0, R)$, and $u = \partial_t u = 0$ on $U \times \{0\}$. Then $u = 0$ in $K(\hat{x}, R)$.

Proof. Fix $t \geq 0$, define

$$E(t) = \frac{1}{2} \int_{B(\hat{x}, R-t)} (|\partial_t u(x,t)|^2 + |\mathrm{D} u(x,t)|^2) \, \mathrm{d}x.$$

Then, if $t \in (0, R)$,

$$\begin{aligned}
\partial_t E(t) &= \frac{1}{2} \left. \frac{\mathrm{d}}{\mathrm{d}h} \right|_{h=0} \left(\int_{B(\hat{x}, R-t-h)} (|\partial_t u(x,t)|^2 + |\mathrm{D}_x u(x,t)|^2) \, \mathrm{d}x \right. \\
&\quad \left. + \int_{B(\hat{x}, R-t)} (|\partial_t u(x,t+h)|^2 + |\mathrm{D}_x u(x,t+h)|^2) \, \mathrm{d}x \right) \\
&= \frac{1}{2} \left(- \int_{\partial B(\hat{x}, R-t)} (|\partial_t u(x,t)|^2 + |\mathrm{D}_x u(x,t)|^2) \, \mathrm{d}S(x) \right. \\
&\quad \left. + \int_{B(\hat{x}, R-t)} (2 \partial_t u \partial_t^2 u + 2 \mathrm{D}u \mathrm{D} \partial_t u) \, \mathrm{d}x \right) \\
&= \int_{\partial B(\hat{x}, R-t)} \left(- \frac{|\partial_t u|^2}{2} - \frac{|\mathrm{D}u|^2}{2} + \partial_t u \mathrm{D}u \cdot \nu \right) \mathrm{d}S(x) \\
&\overset{(*)}{\leq} - \int_{\partial B(x, R-t)} (|\partial_t u| - |\mathrm{D}u|)^2 \, \mathrm{d}S \leq 0,
\end{aligned}$$

where we have used in $(*)$ that

$$(|\partial_t u| - |\mathrm{D}u|)^2 = |\partial_t u|^2 + |\mathrm{D}u|^2 - 2|\partial_t u| \cdot |\mathrm{D}u| \leq |\partial_t u|^2 + |\mathrm{D}u|^2 - 2|\partial_t u| \cdot |\mathrm{D}u \cdot \nu|,$$

because $|\nu| = 1$. It follows that $0 \leq E(t) \leq E(0) = 0$. So, u is constant in $K(\hat{x}, R)$. □

Remark 12.8. Notice that we actually only need u constant in $B(\hat{x}, R)$ at time 0 to obtain that u is constant in $K(\hat{x}, R)$.

Exercise 12.9. Let $\hat{x} \in \mathbb{R}^n$ and $R > 0$. Prove the following uniqueness result: if $u_1, u_2 \in C^2(K(\hat{x}, R))$ are such that $\Box u_1 = \Box u_2$ in $K(\hat{x}, R)$ and $u_1 = u_2$ on $K(\hat{x}, R) \cap \mathbb{R}^n \times \{0\}$, then $u_1 = u_2$. ♢

§12.5. Uniqueness of solution to the wave equation.

Theorem 12.10 (Uniqueness of solution to the wave equation)**.** *Let $U \subset \mathbb{R}^n$ be an open set with C^1 boundary ∂U, and fix $T > 0$. Let $f \in C^0(U_T)$, $g \in C^0(\Gamma_T)$, and $h \in C^0(U)$. Then there is at most one solution in $C^2(\bar{U}_T)$ to*

$$\begin{cases} \Box u = f & \text{in } U_T, \\ u = g & \text{on } \Gamma_T, \\ \partial_t u = h & \text{on } U \times \{0\}. \end{cases} \tag{82}$$

Proof. As usual, if $u_1, u_2 \in C^2(\bar{U} \times [0,T])$ solve (82), then their difference $w = u_2 - u_1$ solve the same problem with $f = 0$, $g = 0$, and $h = 0$. Define

$$E(t) = \int_U (\partial_t w(x,t)^2 + |\nabla w(x,t)|^2) \, \mathrm{d}x.$$

Then

$$\begin{aligned}
E'(t) &= \int_U (2 \partial_t w \partial_t^2 w + 2 \nabla w \nabla \partial_t w) \, \mathrm{d}x \\
&= 2 \int_U \partial_t w \partial_t^2 w + \int_{\partial U} \partial_t w \nabla w \cdot \nu_U \, \mathrm{d}S(x) - \int_U \triangle w \partial_t w \, \mathrm{d}x \\
&= 0,
\end{aligned}$$

because $\partial_t w = 0$on ∂U for all t (because $w = 0$ on $\partial U \times [0,T]$), and because $\partial_t^2 w = \triangle w$ on $U \times (0,T)$. Therefore, E is constant. Since $E(0) = 0$, then $E \equiv 0$. It follows that w is constant, and thus 0 since $w = 0$ on $U \times \{0\}$. □

§12.6. Solution by spherical means: case $n = 1$, d'Alambert's formula. We consider the case $n = 1$. We will solve the PDE

$$\begin{cases} \Box u = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ u = g, \ \partial_t u = h & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{83}$$

with g and h given functions $\mathbb{R} \to \mathbb{C}$. Before we give the full statement that we can prove at the moment, we will see how to find such a formula. So, we forget about the regularity of u, or, in other words, we assume that u is C^∞ (or C^2, which is enough to justify our reasoning).

First, we notice that we can rewrite (83) as

$$(\partial_t + \partial_x)(\partial_t - \partial_x)u = 0.$$

Therefore, if $v = \partial_t u - \partial_x u$, then v solves the transport equation (25) (with $b = 1$):

$$\begin{cases} \partial_t v + \partial_x v = 0 & \text{in } \mathbb{R} \times (0, +\infty), \\ v = h - g' & \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

If u is C^2, then v is C^1 and we know that

$$v(x, t) = v(x - t, 0) = h(x - t) - g'(x - t).$$

Next, u solves $\partial_t u - \partial_x u = v$ in $\mathbb{R} \times (0, +\infty)$. By the solution to the nonhomogeneous transport equation given by Theorem 9.1 (with $b = -1$ and $f = v$), we have

$$\begin{aligned} u(x, t) &= u(x + t, 0) + \int_0^t v(x - (r - t), r)\, \mathrm{d}r \\ &= g(x + t) + \int_0^t (h(x - r + t - r) - g'(x - r + t - r))\, \mathrm{d}r \\ &= g(x + t + \frac{g(x + t - 2r)}{2}\bigg|_0^t + \int_0^t h(x + t - 2r)\, \mathrm{d}r \\ [\bar{r} = x + t - 2r] &= \frac{1}{2}(g(x + t) + g(x - t)) - \int_{x+t}^{x-t} h(\bar{r}) \frac{\mathrm{d}\bar{r}}{2} \\ &= \frac{1}{2}(g(x + t) + g(x - t)) + \frac{1}{2}\int_{x-t}^{x+t} h(r)\, \mathrm{d}r. \end{aligned}$$

We thus obtain *D'Alambert's formula*

$$u(x, t) = \frac{1}{2}(g(x + t) + g(x - t)) + \frac{1}{2}\int_{x-t}^{x+t} h(r)\, \mathrm{d}r. \tag{84}$$

Theorem 12.11 (Solution to $\Box = 0$ for $n = 1$)**.** *Let $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Define $u : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ by D'Alambert's formula* (84)*. Then*

(1) $u \in C^2(\mathbb{R} \times \mathbb{R})$;
(2) $\partial_t^2 u - \partial_x^2 u = \Box u = 0$ in $\mathbb{R} \times \mathbb{R}$;
(3) for every $\hat{x} \in \mathbb{R}$, $u(\hat{x}, 0) = g(\hat{x})$ and $\partial_t u(\hat{x}, 0) = h(\hat{x})$.

Moreover, u is the only solution to (83)*.*

Proof. The proof is left as an exercise. Uniqueness is given by Theorem 12.10. □

Exercise 12.12. Prove Theorem 12.11. ◊

Exercise 12.13. Try to solve the nonhomogeneous version of (83), that is,

$$\begin{cases} \Box u = f & \text{in } \mathbb{R} \times (0, +\infty), \\ u = g, \ \partial_t u = h & \text{on } \mathbb{R} \times \{0\}, \end{cases}$$

for some given f. ◊

Remark 12.14. Solutions to the wave equation given by D'Alambert's formula (84) are of the form

$$u(x,t) = F(x+t) + G(x-t),$$

with $F, G \in C^2(\mathbb{R})$. The two functions represent a forward-moving wave and a backward-moving wave. Can you say which is which?

Remark 12.15. The solutions we have found with D'Alambert's formula (84) extend to negative time!

Remark 12.16. The solutions we have found with D'Alambert's formula (84) are not C^∞ smooth if g and h are not C^∞. This is different from the other equations we have studied so far. Can you spot the difference in the equation? In fact, the highest order part in the Laplace and in the heat equations is the laplacian, while here the whole $\Box$ is homogeneous of order 2.

Remark 12.17. For $n > 1$, we can define $\tilde{u}(x,t) = u(x_1,t)$, where u is a solution in dimension 1 to the wave equation. It follows that $\tilde{u}$ is a solution to the wave equation in $\mathbb{R}^n \times \mathbb{R}$. Therefore, we have found non-smooth solutions to $\Box u = 0$ in all dimensions!

§12.7. A Reflexion method: solution on the half-line. We want to solve the PDE

$$\begin{cases} \Box u = 0 & \text{in } \mathbb{R}_+ \times (0,+\infty), \\ u(x,0) = g(x),\ \partial_t u(x,0) = h(x) & \forall x \in \mathbb{R}_+, \\ u(0,t) = 0 & \forall t > 0. \end{cases} \tag{85}$$

This PDE represent a vibrating string that is pinched at one end and infinite in the other direction. A solution is given in the following Theorem 12.18, whose proof is a direct calculation. The formula (86) is obtained by a *reflection method*, that is, we extend the problem (85) to a PDE on the whole line $\mathbb{R}$ by taking

$$\tilde{u}(x,t) = \begin{cases} u(x,t) & \text{if } x \geq 0, \\ -u(-x,t) & \text{if } x \leq 0; \end{cases}$$

$$\tilde{g}(x) = \begin{cases} g(x) & \text{if } x \geq 0, \\ -g(-x) & \text{if } x \leq 0; \end{cases}$$

$$\tilde{h}(x) = \begin{cases} h(x) & \text{if } x \geq 0, \\ -h(-x) & \text{if } x \leq 0. \end{cases}$$

Then one can convince themselves that $\tilde{u}$ solves the 1D wave equation and thus, we can take $\tilde{u}$ as given from the D'Alambert's formula (84). From there, we can obtain (86).

Theorem 12.18. *Let $\mathbb{R}_+ = (0,+\infty)$. Let $g \in C^2(\bar{\mathbb{R}}_+)$ and $h \in C^1(\bar{\mathbb{R}}_+)$ be such that $g(0) = h(0) = g''(0) = 0$. The function $u : \mathbb{R}_+ \times [0,+\infty) \to \mathbb{C}$,*

$$u(x,t) = \begin{cases} \frac{1}{2}(g(x+t) + g(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} h(y)\,\mathrm{d}y & \text{if } 0 \leq t \leq x, \\ \frac{1}{2}(g(x+t) - g(x-t)) + \frac{1}{2}\int_{-x+t}^{x+t} h(y)\,\mathrm{d}y & \text{if } 0 \leq x \leq t, \end{cases} \tag{86}$$

belongs to $C^2(\mathbb{R}_+ \times (0,+\infty)) \cap C^0(\bar{\mathbb{R}}_+ \times [0,+\infty))$ and it is THE solution to

$$\begin{cases} \Box u = 0 & \text{in } \mathbb{R}_+ \times (0,+\infty), \\ u(x,0) = g(x),\ \partial_t u(x,0) = h(x) & \forall x \in \mathbb{R}_+, \\ u(0,t) = 0 & \forall t > 0. \end{cases} \tag{87}$$

Proof. Left as an exercise. Uniqueness is given by Theorem 12.10.
Hint. The assumption $g''(0) = 0$ implies that $\tilde{g}$ is C^2. □

Exercise 12.19. Prove Theorem 12.18. ◊

Exercise 12.20. For $F, G \in C^2(\mathbb{R})$, define $\tilde{u}(x,t) = F(x+t) + G(x-t)$. Then we know that $\Box\tilde{u}$, see Remark 12.14. For which F and G we have $\tilde{u}(0,t) = 0$ for all t? Solve (87) finding the correct F and G. ◊

Exercise 12.21. Solve the string problem

$$\begin{cases} \Box u = 0 & \text{in } [0,1]\times(0,+\infty), \\ u(x,0)=g(x),\ \partial_t u(x,0)=h(x) & \forall x\in[0,1], \\ u(0,t)=u(1,t)=0 & \forall t>0. \end{cases}$$

You need to find some conditions on g and h to make u of class C^2: it is part of the exercise.

Next, for every $n\in\mathbb{N}$, find $u_n\in C^2([0,1]\times\mathbb{R})$ such that $\Box u_n=0$ and $u_n(k/n,t)=0$ for all $k\in\{1,\dots,n\}$. These functions u_n are called *harmonics* of the string. ♢

§12.8. Spherical means: Euler–Poisson–Darboux equation.

Lemma 12.22 (Euler–Poisson–Darboux equation)**.** *Let $n\ge 2$, $m\ge 2$, $u\in C^m(\mathbb{R}^n\times[0,+\infty))$, $g,h\in C^m(\mathbb{R}^n)$. Suppose*

$$\begin{cases} \partial_t^2 u - \triangle u = 0 & \textit{in } \mathbb{R}^n\times(0,+\infty), \\ u=g,\ \partial_t u = h & \textit{on } \mathbb{R}^n\times\{0\}. \end{cases} \tag{88}$$

Referring to 6, set $U(x;r,t)=\psi_u(t,x;r)$, $G(x;r):=\psi_g(x,r)$, $H(x;r)=\psi_h(x;r)$. Here $\psi_{(\cdot)}$ is defined in (16)*. Then $U(x;\cdot)\in C^m([0,+\infty)\times[0,+\infty))$ and*

$$\begin{cases} \partial_t^2 U - \partial_r^2 U - \frac{n-1}{r}\partial_r U = 0 & \textit{in } (0,+\infty)\times(0,+\infty), \\ U=G,\ \partial_t U = H & \textit{on } (0,+\infty)\times\{0\}. \end{cases} \tag{89}$$

Proof. The regularity of all functions U, G and H is proven in Lemma 6.1. To show (89), we just need to perform the following computations, using again Lemma 6.1.

$$\begin{aligned} \partial_r\psi_u(x,t;r) &= \frac{r}{n}\phi_{\triangle u}(x,t;r) \\ &\overset{(88)}{=} \frac{r}{n}\phi_{\partial_t^2 u}(x,t;r) \\ &= \frac{r}{n}\partial_t^2\phi_u(x,t;r), \\ \partial_r^2\psi_u(x,t;r) &= \frac{1}{n}\partial_t^2\phi_u + \frac{r}{n}\partial_t^2\left(\frac{n}{r}(\psi_u-\phi_u)\right) \\ &= \partial_t^2\psi_u + \frac{1-n}{n}\partial_t^2\phi_u \\ &= \partial_t^2\psi_u + \frac{1-n}{n}\left(\frac{n}{r}\partial_r\psi_u\right) \\ &= \partial_t^2\psi_u + \frac{1-n}{r}\partial_r\psi_u. \end{aligned}$$

□

§12.9. Solution by spherical means: case $n=3$. Here we show how to obtain *Kirchhoff's formula* (90), where we can assume $s=0$ without loss of generality by **§12.2**.

So, let $u\in C^2(\mathbb{R}^3\times[0,+\infty))$ be such that (94) holds, that is

$$\begin{cases} \partial_t^2 u - \triangle u = 0 & \text{in } \mathbb{R}^3\times(0,+\infty), \\ u=g,\ \partial_t u = h & \text{on } \mathbb{R}^3\times\{0\}. \end{cases}$$

Define U, G and H as in Lemma 12.22 and then set

$$\tilde U(x;r,t) = rU(x;r,t), \qquad \tilde G(x;r) = rG(x;r), \qquad \tilde H(x;r) = rH(x;r).$$

Then a direct computation shows that, for each $x\in\mathbb{R}^3$,

$$\begin{cases} \partial_t^2\tilde U - \partial_r^2\tilde U = 0 & \text{for } r,t>0 \\ \tilde U(x;r,0)=\tilde G(x;r),\ \partial_t\tilde U(x;r,0)=\tilde H(x;r) & \text{for } r>0. \end{cases}$$

So, for each $x\in\mathbb{R}^3$, the function $U(x;\cdot,\cdot)$ solves the pinched string problem **§12.7**. Since the solution to the pinched string problem **§12.7** is the only one, we obtain, for $r\in(0,t)$,

$$\tilde U(x;r,t) = \frac12(\tilde G(x;r+t)-\tilde G(x;t-r)) + \frac12\int_{-r+t}^{r+t}\tilde H(x;s)\,\mathrm{d}s.$$

So,

$$\begin{aligned}
u(x,t) &= \lim_{r\to 0} U(x;r,t) = \lim_{r\to 0} \frac{\tilde U(x;r,t)}{r} \\
&= \lim_{r\to 0}\left(\frac{\tilde G(x;r+t) - \tilde G(x;t-r)}{2r} + \fint_{t-r}^{t+r} \tilde H(x;s)\,\mathrm{d}s \right) \\
&= \partial_r \tilde G(x;r)|_{r=t} + \tilde H(x;t) \\
&= (G(x;t) + t\partial_r G(x;r)|_{r=t}) + tH(x;t) \\
&\overset{(19)}{=} \fint_{\partial B(x,t)} g(y)\,\mathrm{d}S(y) + \frac{t^2}{3}\fint_{B(x,t)} \triangle g(y)\,\mathrm{d}y + t\fint_{\partial B(x,t)} h(y)\,\mathrm{d}S(y) \\
&= \fint_{\partial B(x,t)} \left(g(y) + t\nabla g(y)\cdot \frac{y-x}{|y-x|} + th(y)\right) \mathrm{d}S(y) \\
&= \fint_{\partial B(x,t)} (g(y) + \nabla g(y)\cdot (y-x) + th(y))\,\mathrm{d}S(y).
\end{aligned}$$

We have thus obtained the *Kirchhoff's formula.*

Theorem 12.23 (Kirchhoff's formula)**.** *Let $u \in C^2(\mathbb{R}^3 \times [0,+\infty))$ be such that $\Box u = 0$ in $\mathbb{R}^3 \times (0,+\infty)$. Then, for every $s \geq 0$ and every $t > 0$,*

$$u(x,s+t) = \fint_{\partial B(x,t)} (u(y,s) + \nabla_y u(y,s)\cdot(y-x) + t\partial_s u(y,s))\,\mathrm{d}S(y). \tag{90}$$

Vice-versa, let $g \in C^2(\mathbb{R}^3)$ and $h \in C^1(\mathbb{R}^3)$, and define

$$u(x,t) = \fint_{\partial B(x,t)} (g(y) + \nabla_y g(y)\cdot(y-x) + th(y))\,\mathrm{d}S(y). \tag{91}$$

Then $u \in C^2(\mathbb{R}^3 \times [0,+\infty))$ and u is a solution to

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^3 \times (0,+\infty), \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases} \tag{92}$$

Proof. The second part of the theorem, that is, that the function (93) is of class C^2 and that it solves the problem (93), follows from Lemma 6.1 and direct computations.

The first part of the theorem, that is, that every solution to $\Box u = 0$ in $C^2(\mathbb{R}^3 \times [0,+\infty))$ satisfies Kirchhoff's formula (90), is the result of the discussion before the theorem. □

Remark 12.24. Notice that the integral in (90) is supported on the sphere!

§12.10. Solution by spherical means: case $n = 2$.

Remark 12.25. The issue here is that, for $n = 2$, the Euler–Poisson–Darboux equation (89) cannot be transformed into a wave equation. (Why?)

Let $u \in C^2(\mathbb{R}^2 \times [0,+\infty))$ be a solution to the system

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^2 \times (0,+\infty), \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}^2 \times \{0\}. \end{cases}$$

We define $\tilde u \in C^2(\mathbb{R}^3 \times [0,+\infty))$ by $\tilde u(x_1,x_2,x_3,t) = u(x_1,x_2,t)$, $\tilde g \in C^2(\mathbb{R}^3)$ by $\tilde g(x_1,x_2,x_3) = g(x_1,x_2)$, and $\tilde h \in C^1(\mathbb{R}^3)$ by $\tilde h(x_1,x_2,x_3) = u(x_1,x_2)$. Therefore, $\tilde u$ solves

$$\begin{cases} \Box \tilde u = (\partial_t^2 - \triangle)\tilde u = 0 & \text{in } \mathbb{R}^3 \times (0,+\infty), \\ \tilde u = g,\ \partial_t \tilde u = h & \text{on } \mathbb{R}^3 \times \{0\}. \end{cases}$$

To avoid confusion, we denote by B_3 balls in $\mathbb{R}^3$ and by B_2 balls in $\mathbb{R}^2$. Theorem 12.23 tells us that, denoting by $\tilde x = (x_1,x_2,0) \in \mathbb{R}^3$ the lift of the point $x = (x_1,x_2) \in \mathbb{R}^2$, then

$$\begin{aligned}
u(x_1,x_2,t) = \tilde u(x_1,x_2,0,t) &= \fint_{\partial B_3(\tilde x,t)} (g(\tilde y) + \mathrm{D}g(\tilde y)\cdot(\tilde y - \tilde x) + th(\tilde y))\,\mathrm{d}S(\tilde y) \\
&\overset{(14)}{=} \frac{1}{3\omega_3 t^2}\int_{-t}^{t} \int_{\partial B_2(x,\sqrt{t^2-s^2})} (g(y) + th(y) + \mathrm{D}g(y)\cdot(y-x))\,\mathrm{d}S(y) \frac{1}{\sqrt{t^2-s^2}}\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{3\omega_3 t^2}\int_0^t \int_{\partial B_2(x,\sqrt{t^2-s^2})} (g(y)+th(y)+\mathrm{D}g(y)\cdot(y-x))\,\mathrm{d}S(y)\frac{1}{\sqrt{t^2-s^2}}\,\mathrm{d}s \\
[r=\sqrt{t^2-s^2}] &= \frac{2}{3\omega_3 t^2}\int_0^t \int_{\partial B_2(x,r)} (g(y)+th(y)+\mathrm{D}g(y)\cdot(y-x))\,\mathrm{d}S(y)\frac{r}{\sqrt{t^2-r^2}}\frac{\mathrm{d}r}{r} \\
&= \frac{2}{3\omega_3 t^2}\int_0^t \int_{\partial B_2(x,r)} \frac{(g(y)+th(y)+\mathrm{D}g(y)\cdot(y-x))}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}S(y)\,\mathrm{d}r \\
&= \frac{2}{3\omega_3 t^2}\int_{B_2(x,t)} \frac{(g(y)+th(y)+\mathrm{D}g(y)\cdot(y-x))}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}y \\
&= \frac{1}{2}⨍_{B_2(x,t)} \frac{g+th+\mathrm{D}g\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}(y).
\end{aligned}
$$

Theorem 12.26 (Poisson's formula). *Let $u \in C^2(\mathbb{R}^2\times[0,+\infty))$ be such that $\Box u = 0$ in $\mathbb{R}^2\times(0,+\infty)$. Then, for every $s \geq 0$ and every $t>0$,*

$$u(x,s+t) = \frac{1}{2}⨍_{B_2(x,t)} \frac{u(y,s)+t\partial_t(y,s)+\nabla u(y,s)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}(y).$$

Vice-versa, let $g\in C^2(\mathbb{R}^2)$ and $h \in C^1(\mathbb{R}^2)$, and define

$$u(x,t) = \frac{1}{2}⨍_{B_2(x,t)} \frac{g(y)+th(y)+\nabla g(y)\cdot(y-x)}{(t^2-|y-x|^2)^{1/2}}\,\mathrm{d}(y). \tag{93}$$

Then $u\in C^2(\mathbb{R}^2\times[0,+\infty))$ and u is a solution to

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^2\times(0,+\infty),\\ u=g,\ \partial_t u = h & \text{on } \mathbb{R}^2\times\{0\}. \end{cases} \tag{94}$$

Exercise 12.27. Compute ω_3, the volume of the unit ball in $\mathbb{R}^3$.

Solution:

$$
\begin{aligned}
\omega_3 &= 2\int_{B_2(x,1)} \sqrt{1-|x|^2}\,\mathrm{d}x \\
&= 2\int_0^1\int_0^{2\pi} \sqrt{1-r^2}\,\mathrm{d}\theta r\,\mathrm{d}r \\
&= 4\pi\int_0^1 r\sqrt{1-r^2}\,\mathrm{d}r \\
[r=\sin t] &= 4\pi\int_0^{\pi/2} \sin t\cos t\,\mathrm{d}t = [...] = \frac{4\pi}{3}.
\end{aligned}
$$

◊

§12.11. Solution of the wave equation in all dimensions. For the next theorem, see [7, Th. 5.15, page 170] or [5, Th. 2.4.2, page 77].

Theorem 12.28 (Odd dimensions). *Let $n\geq 3$ odd, say $n=2m-1$ for $m\geq 2$, or $m=\frac{n+1}{2}$. Let $g\in C^{m+1}(\mathbb{R}^n)$, $h\in C^m(\mathbb{R}^n)$, and define*

$$
\begin{aligned}
u(x,t) = \frac{1}{\gamma_n}\Bigg[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} &\left(t^{n-2}⨍_{\partial B(x,t)} g(y)\,\mathrm{d}S(y)\right) \\
&+ \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2}⨍_{\partial B(x,t)} h(y)\,\mathrm{d}S(y)\right)\Bigg],
\end{aligned} \tag{95}
$$

where γ_n is the product of the odd numbers from 1 to $n-2$.

Then $u\in C^2(\mathbb{R}^n\times[0,+\infty)$ and u is a solution to

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^n\times(0,+\infty),\\ u=g,\ \partial_t u = h & \text{on } \mathbb{R}^n\times\{0\}. \end{cases}$$

For the next theorem, see [7, Th. 5.17, page 171] or [5, Th. 2.4.3, page 80].

Theorem 12.29 (Even dimensions)**.** *Let $n \geq 2$ even, say $n = 2m - 2$ for $m \geq 2$, or $m = \frac{n+2}{2}$. Let $g \in C^{m+1}(\mathbb{R}^n)$, $h \in C^m(\mathbb{R}^n)$, and define*

(96)

$$u(x,t) = \frac{1}{\beta_n}\Bigg[\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{g(y)}{(t^2 - |y-x|^2)^{1/2}}\,\mathrm{d}S(y)\right) + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{h(y)}{(t^2 - |y-x|^2)^{1/2}}\,\mathrm{d}S(y)\right)\Bigg],$$

where β_n is the product of the even numbers from 1 to n.

Then $u \in C^2(\mathbb{R}^n \times [0,+\infty)$ and u is a solution to

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Remark 12.30. Notice that in both theorems 12.28 and 12.29,

$$m = \left\lfloor \frac{n}{2} \right\rfloor + 1.$$

Exercise 12.31. Recover Kirchhoof's formula (93) from (95). ♢

Exercise 12.32. Recover Poisson's formula (93) from (96). ♢

§12.12. Solution to the nonhomogeneous wave equation: Duhamel's principle. For the next theorem, see [7, Th. 5.25, page 175] or [5, Th. 2.4.4, page 81].

Theorem 12.33 (Nonhomogeneous equation with null initial data)**.** *Let $n \geq 2$ and $f \in C^{\lfloor \frac{n}{2} \rfloor + 1}(\mathbb{R}^n \times [0,+\infty))$. For every $s > 0$, let $u_s : \mathbb{R}^n \times [s,+\infty) \to \mathbb{C}$ be the solution in $C^2(\mathbb{R}^n \times [0,+\infty))$ to*

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = 0 & \text{in } \mathbb{R}^n \times (s,+\infty), \\ u = 0,\ \partial_t u = f(\cdot, s) & \text{on } \mathbb{R}^n \times \{s\}. \end{cases}$$

Define $u : \mathbb{R}^n \times [0,+\infty) \to \mathbb{C}$ by

$$u(x,t) = \int_0^t u_s(x,t)\,\mathrm{d}s.$$

Then $u \in C^2(\mathbb{R}^n \times [0,+\infty))$ and u is a solution to

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = f & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u = 0,\ \partial_t u = 0 & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Exercise 12.34. Prove Theorem 12.33. ♢

Exercise 12.35. Write explicitly u from Theorem 12.33 for $n = 2$ and $n = 3$. ♢

Theorem 12.36 (Nonhomogeneous wave equation)**.** *Let $n \geq 2$ and $m = \lfloor \frac{n}{2} \rfloor + 1$. Let $f \in C^m(\mathbb{R}^n \times [0,+\infty))$, Let $g \in C^{m+1}(\mathbb{R}^n)$, and $h \in C^m(\mathbb{R}^n)$.*

Let u_0 be the function given by Theorem 12.28 and 12.29, and u_1 the function given by Theorem 12.33. Set $u = u_0 + u_1$. Then $u \in C^2(\mathbb{R}^n \times [0,+\infty))$ and u is a solution to

$$\begin{cases} \Box u = (\partial_t^2 - \triangle)u = f & \text{in } \mathbb{R}^n \times (0,+\infty), \\ u = g,\ \partial_t u = h & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Exercise 12.37. Prove Theorem 12.36. ♢

Part 2. Distributions

13. Distributions

Here we run through the fundamentals of the theory of distributions omitting a few details. The details can be recovered by thinking through this material, or reading Rudin's book [11], which I strongly recommend:
• W. Rudin. *Functional analysis.* Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424

Another valuable reference is Hörmander's first book of his series on linear partial differential operators [9]:
• L. Hörmander. *The analysis of linear partial differential operators. I.* vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1983, pp. ix+391
Hörmander monograph is one of the most important resources on PDE. The text is has a very high density, and this can lead to obscurity: if you read it slowly, it will become crystalline clear!

An even denser reference is Chapter Four of Federer's book [6]:
• H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969, pp. xiv+676
There, you can find a more general construction of distributions, where test functions are smooth functions $\Omega \to Y$, where Ω is an open set in a Banach space and Y is another Banach space. As for Hörmander, Federer's style is at times obscure, but extremely precise, abstract and general. Read it slowly.

Never forget to check out Wikipedia:
• `https://en.wikipedia.org/wiki/Distribution_(mathematics)`

Finally, the founding father of distributional calculus was Laurent Schwartz[3], who won the Fields medal in 1950 for the reason[4]:

> Developed the theory of distributions, a new notion of generalized function motivated by the Dirac delta-function of theoretical physics.

He then wrote a beautiful autobiography [12], which I suggest everyone to read:
• L. Schwartz. *A mathematician grappling with his century. Transl. from the French by Leila Schneps.* English. Basel: Birkhäuser, 2001.
This is the English translation. I have an Italian translation at home, but the original is in French. Probably, there is a German translation too.

§13.1. Test Functions. For every set $E \subset \mathbb{R}^n$, we define

$$\mathscr{D}(E) = \{\phi \in C_c^\infty(\mathbb{R}^n) : \mathrm{spt}(\phi) \subset E\}.$$

If $\Omega \subset \mathbb{R}^n$ is an open set, then $\mathscr{D}(\Omega)$ is the space $C_c^\infty(\Omega)$ of smooth functions $\Omega \to \mathbb{C}$ with compact support. We call elements of $\mathscr{D}(\Omega)$ *test functions.* We write just $\mathscr{D}$ for $\mathscr{D}(\mathbb{R}^n)$.

We endow $\mathscr{D}(\Omega)$ with a topology that has the following property: For every sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$,

$$\phi_j \overset{\mathscr{D}(\Omega)}{\longrightarrow} \phi \quad \Leftrightarrow \quad \begin{array}{c} \exists K \Subset \Omega\ \forall j \in \mathbb{N}\ \mathrm{spt}(\phi_j) \subset K, \text{ and} \\ \forall \alpha \in \mathbb{N}^n\ \lim_{j\to\infty} \|\mathrm{D}^\alpha \phi_j - \mathrm{D}^\alpha \phi\|_{L^\infty} = 0. \end{array} \tag{97}$$

Exercise 13.1. We can see $\mathscr{D}(\Omega)$ as a subspace of $\mathscr{D}(\mathbb{R}^n)$, but not as a closed subspace. Why? ◊

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[3] `https://en.wikipedia.org/wiki/Laurent_Schwartz`
[4] `https://www.mathunion.org/fileadmin/IMU/Prizes/Fields/1950/index.html`

§13.2. The topology of test functions. We will use only the In most of the situations, Property (97) is everything we need to know about the topology of $\mathscr{D}(\Omega)$. However, the fact that this notion of convergence descends from a topology, is a non-trivial fact which needs precise definition of that topology.

There are three ways to construct the topology of test functions. First of all, for $K \subset \mathbb{R}^n$ compact, the space $\mathscr{D}(K)$ is a Frechét space when endowed with the family of pseudonorms

$$\|u\|_\alpha = \|\mathrm{D}^\alpha u\|_{L^\infty(K)}, \qquad \alpha \in \mathbb{N}^n.$$

The first way to construct the topology of $\mathscr{D}(\Omega)$ is defining the collection β of all convex balanced sets $W \subset \mathscr{D}(\Omega)$ such that $\mathscr{D}(K) \cap W$ is open $\mathscr{D}(K)$ for all $K \subset \Omega$ compact. A a set W is *balanced* if $\lambda W \subset W$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$. The collection β induces a topology τ make of unions of elements of $\{x + W : x \in \mathscr{D}(K),\ w \in \beta\}$. Then τ makes $\mathscr{D}(\Omega)$ into a locally convex topological vector space.

The other two ways are in terms of initial and final topologies:

Definition 13.2 (Initial, or projective, topology)**.** Given a set Y and a family of topological spaces $\{Z_i\}_{i\in I}$ and functions $f_i : Y \to Z_i$. The *initial topology* or *projective topology* induced by the family of functions f_i is the coarsest (i.e., smallest) topology in Y that makes all functions f_i continuous.

Definition 13.3 (Final, or inductive, topology)**.** Given a set Y and a family of topological spaces $\{X_i\}_{i\in I}$ and functions $f_i : X_i \to Y$. The *final topology* or *inductive topology* induced by the family of functions f_i is the finest (i.e., largest) topology in Y that makes all functions f_i continuous.

So, we start with the Banach spaces

$$C_c^m(K) = \{\phi : \mathbb{R}^n \to \mathbb{C} \text{ of class } C^m \text{ with } \operatorname{spt}(\phi) \subset K\},$$

where the norm is

$$\|\phi\|_{C^m(\Omega)} = \max\{\|\mathrm{D}^\alpha \phi\|_{L^\infty(\Omega)} : |\alpha| \le m\}. \tag{98}$$

Then we set

$$C_c^\infty(K) = \bigcap_{m\in\mathbb{N}} C_c^m(K)$$

endowed with the initial topology induced by the functions $C_c^\infty(K) \hookrightarrow C_c^m(K)$; and then

$$\mathscr{D}(\Omega) = C_c^\infty(\Omega) = \bigcup_{K\Subset\Omega} C_c^\infty(K) = \bigcup_{K\Subset\Omega} \bigcap_{m\in\mathbb{N}} C_c^m(K), \tag{99}$$

endowed with the final topology induced by the functions $C_c^\infty(K) \hookrightarrow C_c^\infty(\Omega)$.

An equivalent way is to take first

$$C_c^m(\Omega) = \bigcup_{K\Subset\Omega} C_c^m(K)$$

endowed with the final topology induced by the functions $C_c^m(K) \hookrightarrow C_c^m(\Omega)$, and then

$$\mathscr{D}(\Omega) = C_c^\infty(\Omega) = \bigcap_{m\in\mathbb{N}} C_c^m(\Omega) = \bigcap_{m\in\mathbb{N}} \bigcup_{K\Subset\Omega} C_c^m(K) \tag{100}$$

endowed with the initial topology induced by the functions $C_c^\infty(\Omega) \hookrightarrow C_c^m(\Omega)$.

Exercise 13.4. Show that the two topologies coming from (99) and (100) are the same. ◊

§13.3. Continuity of linear operators.

Proposition 13.5. *Let Y be a locally convex space and $L : \mathscr{D}(\Omega) \to Y$ linear. Then the following are equivalent:*

(1) L is continuous;
(2) if $\phi_j \to 0$ in $\mathscr{D}(\Omega)$ then $L\phi_j \to 0$ in Y;
(3) the restrictions of L to every $C_c^\infty(K) \subset \mathscr{D}(\Omega)$, for $K \Subset \Omega$, are continuous.

Proof. The equivalence (2) ⇔ (3) is clear, as it is clear (1) ⇒ (2). The implication (3) ⇒ (1) follows from the properties of the topology in $\mathscr{D}(\Omega)$. See Rudin [11, Thm.6.6]. □

§13.4. Distributions. A distribution is an element of the dual space $\mathscr{D}'(\Omega)$, that is, a continuous linear functional $\mathscr{D}(\Omega) \to \mathbb{C}$. The topology of $\mathscr{D}'(\Omega)$ is the weak* topology: For every sequence $\{A_j\}_{j\in\mathbb{N}} \subset \mathscr{D}'(\Omega)$ and $A \in \mathscr{D}(\Omega)$,

$$A_j \overset{\mathscr{D}'(\Omega)}{\longrightarrow} A \quad \Leftrightarrow \quad \forall \phi \in \mathscr{D}(\Omega) \ \lim_{j\to\infty} A_j[\phi] = A[\phi].$$

We write just $\mathscr{D}'$ for $\mathscr{D}'(\mathbb{R}^n)$.

§13.5. Functions as distributions. To every $f \in L^1_{\text{loc}}(\Omega)$, we associate a distribution $A_f \in \mathscr{D}'(\Omega)$ defined by, for $\phi \in \mathscr{D}(\Omega)$,

$$A_f[\phi] = \int_\Omega f(x)\phi(x)\,\mathrm{d}x.$$

Let's show that A_f is a distribution. Clearly A_f is linear. We need to show it is continuous. If $\phi_j \to \phi$ in $\mathscr{D}(\Omega)$, then there is $K \Subset \Omega$ with $\operatorname{spt}(\phi_j) \subset K$ and $\|\phi_j - \phi\|_{L^\infty(\Omega)} \to 0$. Therefore,

$$|A_f[\phi_j] - A_f[\phi]| \le \int_\Omega |f(x)||\phi_j(x) - \phi(x)|\,\mathrm{d}x \le \int_K |f(x)|\,\mathrm{d}x \|\phi_j - \phi\|_{L^\infty(\Omega)} \to 0.$$

This shows that A_f is continuous.

The Fundamental Theorem of Calculus implies that, if $f, g \in L^1_{\text{loc}}(\Omega)$ are such that $A_f = A_g$ as distributions, then $f = g$ almost everywhere, that is, $f = g$ in $L^1_{\text{loc}}(\Omega)$. We thus have an inclusion $L^1_{\text{loc}}(\Omega) \hookrightarrow \mathscr{D}(\Omega)$.

We can say more about this inclusion: it is continuous. Here we consider on $L^1_{\text{loc}}(\Omega)$ the topology of local convergence in L^1, that is, $f_j \to f$ in $L^1_{\text{loc}}(\Omega)$ if and only if, for every $K \Subset \Omega$, we have $f_j|_K \to f|_K$ in $L^1(K)$, i.e., $\|f_j - f\|_{L^1(K)} \to 0$. So, if $f_j \to f$ in $L^1_{\text{loc}}(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$, $|A_{f_j}[\phi] - A_f[\phi]| \le \int_{\operatorname{spt}(\phi)} |f_j(x) - f(x)||\phi(x)|\,\mathrm{d}x \le \|f_j - f\|_{L^1(\operatorname{spt}(\phi))}\|\phi\|_{L^\infty} \to 0$.

Exercise 13.6. Find a sequence $f_j \in L^1_{\text{loc}}(\mathbb{R})$ such that $\|f_j\|_{L^1([0,1])} = 1$ but $A_{f_j} \to 0$ in $\mathscr{D}'(\mathbb{R})$.

Hint: Take $f_j(x) = \sum_{j=1}^{2^n} (-1)^j \mathbb{1}_{((j-1)/2^n, j/2^n)}(x)$. Then $\int_0^1 |f_j(x)|\,\mathrm{d}x = 1$. If $\phi \in \mathscr{D}(\Omega)$, then there is L such that $|\phi(x) - \phi(y)| < L|x - y|$ for every $x, y \in \mathbb{R}$.

$$\begin{aligned}
\left|\int_\mathbb{R} f_j(x)\phi(x)\,\mathrm{d}x\right| &= \left|\sum_{j=1}^{2^n} (-1)^j \int_{(j-1)/2^n}^{j/2^n} \phi(x)\,\mathrm{d}x\right| \\
&= \left|\sum_{j=1}^{2^{n-1}} \int_0^{1/2^n} \left(-\phi\left(\frac{j-1}{2^{n-1}} + x\right) + \phi\left(\frac{j-1}{2^{n-1}} + \frac{1}{2^n} + x\right)\right)\mathrm{d}x\right| \\
&\le \sum_{j=1}^{2^{n-1}} \int_0^{1/2^n} \frac{L}{2^n}\,\mathrm{d}x \\
&= \frac{L}{2^n}\frac{2^{n-1}}{2^n} = \frac{L}{2^{n+1}}.
\end{aligned}$$

◊

Exercise 13.7. Show that $f_j \to 0$ weakly* in $L^1_{\text{loc}}(\mathbb{R}^n)$, if and only if $A_{f_j} \to 0$ in $\mathscr{D}'(\mathbb{R}^n)$. The weak* convergence is $\int_\mathbb{R} f_j g\,\mathrm{d}x \to 0$ for all $g \in L^\infty(\mathbb{R}^n)$ with compact support. ◊

After this, we can denote still by f the distribution A_f, that is,

$$L^1_{\text{loc}}(\Omega) \subset \mathscr{D}'(\Omega).$$

Exercise 13.8. Let $\{u_k\}_{k\in\mathbb{N}} \subset C^\infty(\Omega)$ be a sequence of harmonic functions and suppose that $u_k \to A$ in $\mathscr{D}'(\Omega)$. Show that A is a harmonic function. ◊

§13.6. Measures as distributions. Let μ be a Radon measure on Ω: $\mu \in \texttt{Rad}(\Omega)$. Then we define the distribution

$$A_\mu[\phi] = \int_\Omega \phi(x)\,\mathrm{d}\mu(x).$$

As we saw with functions, the map $\mu \mapsto A_\mu$ is a continuous embedding:

$$\texttt{Rad}(\Omega) \subset \mathscr{D}'(\Omega).$$

Examples of Radon measures:

(1) The *Dirac delta* centered at $x \in \Omega$ is the measure δ_x defined by: $\delta_x(E) = 1$ if $x \in E$, $\delta_x(E) = 0$ if $x \notin E$. On test functions, the Dirac delta acts as an evaluation: $\delta_x[\phi] = \phi(x)$.
(2) If $E \subset \Omega$ closed or open, the measure $\mathscr{L}^n|_E$ is Radon.
(3) Integration over embedded submanifolds of Ω are Radon measures.

§13.7. Order of a distribution. A distribution $A \in \mathscr{D}'(\Omega)$ *has order (up to)* N is there is $C < \infty$ with

$$\forall \phi \in \mathscr{D}(\Omega), \qquad A[\phi] \le C\|\phi\|_{C^N(\Omega)}.$$

Recall that the norm $\|\phi\|_{C^N(\Omega)}$ was defined in (98). Notice that if A has order N, then it has also order $N+1$. We say that A has order exactly N if it has order N but not order $N-1$.

Proposition 13.9. *A linear functional $A : \mathscr{D}'(\Omega) \to \mathbb{C}$ is continuous, i.e., a distribution, if and only if it has locally finite order, that is, for every $K \Subset \Omega$ there are $N \in \mathbb{N}$ and $C < \infty$ such that, for every $\phi \in \mathscr{D}(\Omega)$ with* $\mathrm{spt}(\phi) \subset K$*, $A[\phi] \le C\|\phi\|_{C^N(\Omega)}$.*

Proof. In this proof, we use two key facts. First, the norms defined in (98) satisfy $\|\phi\|_{C^N(\Omega)} \ge \|\phi\|_{C^k(\Omega)}$ whenever $k \le N$. Second, if we fix $k \in \mathbb{N}$ and $K \Subset \Omega$, then A is a continuous linear functional $(\mathscr{D}(K), \|\cdot\|_{C^k(K)}) \to \mathbb{C}$, that is, there is C_k such that $A[\phi] \le C_k\|\phi\|_{C^k(\Omega)}$ for every $\phi \in \mathscr{D}(K)$. This follows from the very definition of distribution.

So, arguing by contradiction, assume our proposition is false, that is, there is $K \Subset \Omega$ such that, for every $N \in \mathbb{N}$ there is $\phi_N \in \mathscr{D}(\Omega)$ with $\mathrm{spt}(\phi_N) \subset K$, and $A[\phi_N] \ge N\|\phi_N\|_{C^N(K)}$. We have reached a contradiction: for every $N \ge k$, we should also have $N\|\phi_N\|_{C^k(\Omega)} \le N\|\phi_N\|_{C^N(\Omega)} \le A[\phi_N] \le C_k\|\phi_N\|_{C^k(\Omega)}$. Since $A[\phi_N] \ne 0$, then $\|\phi_N\|_{C^k(\Omega)} \ne 0$, and thus we get $N \le C_k$ for all $N \ge k$: $\lightning$.

We conclude that such a sequence $\{\phi_N\}_N$ cannot exists. □

Exercise 13.10. Show that, if $A \in \mathscr{D}'(\Omega)$ has finite order N, then A extends as a continuous linear operator from $\mathscr{D}(\Omega)$ to $C^N(\Omega)$. ♢

§13.8. Distributions of order 0. If X is a topological space, we define $C_c(X)$ as the space of continuous functions $X \to \mathbb{C}$ with compact support endowed with the L^∞ norm. The closure of $C_c(X)$ in $C(X)$ is $C_0(X)$, which is the space of continuous functions vanishing "at infinity", that is,

$$C_0(X) = \{f \in C(X) : \text{for every } \epsilon > 0 \text{ the set } \{|f| \ge \epsilon\} \text{ is compact}\}.$$

The space $C_0(X)$ is a Banach space when endowed with the L^∞ norm. Its (topological) dual $C_0(X)'$ is also a Banach space when endowed with the operator norm $\|\xi\|_{C_0(X)'} = \sup\{\xi[\phi] : \phi \in C_0(X),\ \|\phi_{L^\infty} \le 1\}$.

We define $\texttt{Rad}(X;\mathbb{C})$ as the space of all $\mathbb{C}$-valued Radon measures: the precise definition goes as follows. Let $\mathscr{B}(X)$ be the σ-algebra of all Borel sets. A *positive Radon measure* on X is a measure $\lambda : \mathscr{B}(X) \to [0, +\infty]$ such that (see [8, page 212]):

(1) λ is outer regular on all Borel sets, that is, if $E \in \mathscr{B}(X)$, then

$$\lambda(E) = \inf\{\lambda(U) : E \subset U,\ U \text{ open}\};$$

(2) λ is inner regular on all open sets, that is, if $E \subset X$ is open, then

$$\lambda(E) = \sup\{\lambda(K) : K \subset E,\ K \text{ compact}\};$$

(3) λ is finite on compact sets, that is, $\lambda(K) < \infty$ for all $K \subset X$ compact.

We denote by $\mathtt{Rad}(X;[0,+\infty])$ the space of all positive Radon measures on X. A complex-valued or real-valued Radon measure is a Borel measure $\mu : \mathscr{B}(X) \to \mathbb{K}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ whose total variation is $|\mu| \in \mathtt{Rad}(X;[0,+\infty])$. We call the space of these measures $\mathtt{Rad}(X;\mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. In fact, if $\mu \in \mathtt{Rad}(X;\mathbb{K})$, then $|\mu|(X) < \infty$. Moreover, if $\mu \in \mathtt{Rad}(X;\mathbb{R})$, then there are $\mu_+, \mu_- \in \mathtt{Rad}(X;[0,+\infty))$ such that $\mu = \mu_+ - \mu_-$. If $\mu \in \mathtt{Rad}(X;\mathbb{C})$, then there are $\mu_r, \mu_i \in \mathtt{Rad}(X;\mathbb{R})$ such that $\mu = \mu_r + i\mu_i$.

Recall the following result (see [8, Thm 7.17, page 223])

Theorem 13.11 (Riesz Representation Theorem)**.** *Let X be a topological space that is locally compact*[5] *and Hausdorff*[6]*. For $\mu \in \mathtt{Rad}(X;\mathbb{C})$ and $f \in C_0(X)$, define $I_\mu(f) = \int_X f \,\mathrm{d}\mu$. Then I_μ is an element of the dual $C_0(X)'$ and the map $\mu \mapsto I_\mu$ is an isometric equivalence of $\mathtt{Rad}(X;\mathbb{C})$ with $C_0(X)'$.*

Proposition 13.12. *A distribution $A \in \mathscr{D}'(\Omega)$ has order zero if and only if it is a Radon measure.*

Proof. We already know that Radon measures are distributions of order zero. Let's prove the other implication.

If $A \in \mathscr{D}'(\Omega)$ has order zero, then there is C such that $A[\phi] \leq C\|\phi\|_{L^\infty}$ for all $\phi \in \mathscr{D}(\Omega)$. Since $\mathscr{D}(\Omega)$ is dense in $C_0(\Omega)$, it follows that A continuously extends to a linear operator $C_0(\Omega) \to \mathbb{C}$. By Riesz Representation Theorem 13.11, there is a Radon measure $\mu \in \mathtt{Rad}(\Omega;\mathbb{C})$ such that $A = I_\mu$. This means for us that A is a Radon measure. □

Exercise 13.13. Show that $\mathscr{D}(\Omega)$ is dense in $(C_0(\Omega), \|\cdot\|_{L^\infty})$. ◊

§13.9. Distributions of order 1. Here are some examples of distributions of order 1.

(1) $A[\phi] = \partial_x \phi(0)$;
(2) $A[\phi] = \int_{\partial B(x,r)} \nabla\phi(y) \cdot \frac{(y-x)}{r} \,\mathrm{d}S(y)$.
(3) In general, if $\Sigma \subset \mathbb{R}^n$ is a smooth submanifold, $v : \Sigma \to \mathbb{R}^n$ a smooth vector field, and $\mathrm{d}S$ is the surface measure on Σ, then $A[\phi] = \int_\Sigma \nabla\phi(y) \cdot v(y) \,\mathrm{d}S(y)$ is a distribution of order 1.
(4) For example,
$$A_{x,r}[\phi] = \fint_{\partial B(x,r)} \nabla\phi(y) \cdot (y-x) \,\mathrm{d}y.$$
(5) the distribution $A[\phi] = \int_0^1 \phi'(x) \,\mathrm{d}x$ is a distribution of order one on $\mathbb{R}$: $|A[\phi]| \leq \int_0^1 |\phi'(x)| \,\mathrm{d}x \leq \|\phi\|_{C^1}$. However, A is actually of order zero: $|A[\phi]| = |\phi(1) - \phi(0)| \leq 2\|\phi\|_{C^0}$.

§13.10. Principal value. The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$, $f(x) = \frac{1}{x}$, is not integrable in a neighborhood of 0. For this reason, we cannot see it as a distribution on $\mathbb{R}$, although it is a distribution on $\mathbb{R} \setminus \{0\}$, because $f \in L^1_{\mathrm{loc}}(\mathbb{R} \setminus \{0\})$. However, we can define a distribution on $\mathbb{R}$ as follows:

$$\mathscr{D}(\mathbb{R}) \ni \phi \mapsto \lim_{\epsilon \to 0} \int_{\mathbb{R} \setminus [-\epsilon,\epsilon]} \frac{\phi(x)}{x} \,\mathrm{d}x = \mathrm{p.v.} \int_{\mathbb{R}} \frac{\phi(x)}{x} \,\mathrm{d}x \tag{101}$$

If $\mathrm{spt}(\phi) \subset [-a,a]$, then

$$\int_{\mathbb{R} \setminus [-\epsilon,\epsilon]} \frac{\phi(x)}{x} \,\mathrm{d}x = \int_{[-a,a] \setminus [-\epsilon,\epsilon]} \frac{\phi(x) - \phi(0)}{x} \,\mathrm{d}x.$$

Since $\left| \frac{\phi(x)-\phi(0)}{x} \right| \leq \|\phi'\|_{L^\infty}$, then the latter integral converges as $\epsilon \to 0$:

$$\mathrm{p.v.} \int_{\mathbb{R}} \frac{\phi(x)}{x} \,\mathrm{d}x = \lim_{\epsilon \to 0} \int_{[-a,a] \setminus [-\epsilon,\epsilon]} \frac{\phi(x) - \phi(0)}{x} \,\mathrm{d}x = \int_{[-a,a]} \frac{\phi(x) - \phi(0)}{x} \,\mathrm{d}x.$$

So, (101) defines a distribution by Proposition 13.9.

---

[5]locally compact: for every $x \in X$ and every $U \subset X$ open with $x \in U$ there exists $V \subset U$ compact with $x \in \mathrm{interior}(K)$

[6]Hausdorff: for every $x, y \in X$ distinct there are $U, V \subset X$ open such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$.

§13.11. **Adjoint operators.**

Proposition 13.14. *Let Ω_1 and Ω_2 be open subsets of $\mathbb{R}^n$. Let $\Phi : \mathscr{D}(\Omega_1) \to \mathscr{D}(\Omega_2)$ be a continuous linear operator. Then there is a sequentially*[7] *continuous linear operator $\Phi^* : \mathscr{D}'(\Omega_2) \to \mathscr{D}'(\Omega_1)$ such that, for every $A \in \mathscr{D}'(\Omega_2)$ and $\phi \in \mathscr{D}(\Omega_1)$,*

$$\Phi^* A[\phi] = A[\Phi\phi]. \tag{102}$$

Proof. For every $A \in \mathscr{D}'(\Omega_2)$, define $\Phi^* A$ by (102). Notice that, $\Phi^* A = A \circ \Phi$. Therefore, since both A and Φ are linear and continuous, then $\Phi^* A$ is linear and continuous. We need to show that Φ^* is continuous. Let $A_j \to A$ in $\mathscr{D}'(\Omega_2)$. Then, for every $\phi \in \mathscr{D}(\Omega_1)$, we have $\lim_{j\to\infty} \Phi^* A_j[\phi] = \lim_{j\to\infty} A_j[\Phi\phi] = A[\Phi\phi] = \Phi^* A[\phi]$. This shows that Φ^* is sequentially continuous. □

Proposition 13.14, combined with Proposition 13.5, is the key tool to extend operations from functions to distributions. We will use it all the time!

§13.12. **Derivatives of distributions.** If $A \in \mathscr{D}'(\Omega)$ and $\alpha \in \mathbb{N}^n$, define, for every $\phi \in \mathscr{D}(\Omega)$,

$$\mathrm{D}^\alpha A[\phi] = (-1)^{|\alpha|} A[\mathrm{D}^\alpha \phi]. \tag{103}$$

Exercise 13.15. Show that, if $\alpha \in \mathbb{N}^n$, the function $\phi \mapsto \mathrm{D}^\alpha \phi$ is a continuous linear operator $\mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$. ◊

By Exercise 13.15 and Proposition 13.14, the $A \mapsto \mathrm{D}^\alpha A$ is a continuous operator $\mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$.

Exercise 13.16. Let $f \in C^N(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$. Show that, for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N$,

$$\int_\Omega \mathrm{D}^\alpha f(x)\phi(x)\,\mathrm{d}x = (-1)^{|\alpha|} \int_\Omega f(x) \mathrm{D}^\alpha \phi(x)\,\mathrm{d}x.$$

In other words, $\mathrm{D}^\alpha A_f = A_{\mathrm{D}^\alpha f}$. ◊

Exercise 13.17. Show that, if $A \in \mathscr{D}'(\Omega)$, then $\mathrm{D}^\alpha \mathrm{D}^\beta A = \mathrm{D}^{\alpha+\beta} A = \mathrm{D}^\beta \mathrm{D}^\alpha A$ for all $\alpha, \beta \in \mathbb{N}^n$. ◊

Distributions are thus infinitely differentiable: this is one of the main features of distributions. In particular, we have derivatives of every order for each function in $L^1_{\mathrm{loc}}(\Omega)$. Exercise 13.16 shows that, if a distribution A_f is a function f of class C^N, then derivatives defined by (103) are coherent with derivatives of f.

Proposition 13.18. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ be such that there is $g \in L^1(\mathbb{R})$ with $\mathrm{D}A_f = A_g$, i.e., $\mathrm{D}f = g$ in distributional sense. Then, for almost every $x \in \mathbb{R}$,*

$$f(x) = \int_{-\infty}^x g(y)\,\mathrm{d}y.$$

Consequently, up to changing f on a set of measure zero, f is absolutely continuous and $f' = g$. In particular, if f and g are continuous, then $f \in C^1(\mathbb{R})$ and $f' = g$.

Proof. The identity $\mathrm{D}A_f = A_g$ means that, for every $\phi \in \mathscr{D}(\mathbb{R})$,

$$\int_{-\infty}^\infty g(x)\phi(x)\,\mathrm{d}x = -\int_{-\infty}^\infty f(x)\phi'(x)\,\mathrm{d}x. \tag{104}$$

[7] Recall that $T : X \to Y$ is *sequentially continuous* if for every $x_j \to x_\infty$ in X we have $Tx_j \to T_x\infty$ in Y. Instead, T is (topologically) continuous if for every open set $V \subset Y$ the preimage $T^{-1}(V)$ is open in X. In topological spaces, (topological) continuity implies sequential continuity. The converse is false in this generality.

(A way to recover continuity from convergence is by means of *nets*, a generalization of sequences: a *sequence in* X is a function $\mathbb{N} \to X$, a *net in* X is a function $\omega \to X$ for some ordinal ω. "Netial" continuity implies continuity.)

It remains unclear to me if sequential continuity implies continuity in the context of Proposition **§13.11**.

Define

$$v(x) = \int_{-\infty}^{x} g(y)\,\mathrm{d}y.$$

We claim that $A_f = A_v$. If $\phi \in \mathscr{D}(\mathbb{R})$, then

$$\begin{aligned}
A_v[\phi] &= \int_{-\infty}^{\infty} v(x)\phi(x)\,\mathrm{d}x \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{x} g(y)\phi(x)\,\mathrm{d}y\,\mathrm{d}x \\
[\text{Change of variables}] &= \int_{-\infty}^{\infty}\int_{0}^{\infty} g(t)\phi(t+s)\,\mathrm{d}s\,\mathrm{d}t \\
&= \int_{0}^{\infty}\int_{-\infty}^{\infty} g(t)\phi(t+s)\,\mathrm{d}t\,\mathrm{d}s \\
&\overset{(104)}{=} -\int_{0}^{\infty}\int_{-\infty}^{\infty} f(t)\phi'(t+s)\,\mathrm{d}t\,\mathrm{d}s \\
&= -\int_{-\infty}^{\infty} f(t)(-\phi(t))\,\mathrm{d}t = A_f[\phi].
\end{aligned}$$

where we performed the following change of variables: $x = t + s$, $y = t$, $\mathrm{d}x \wedge \mathrm{d}y = (\mathrm{d}t + \mathrm{d}s) \wedge \mathrm{d}t = \mathrm{d}s \wedge \mathrm{d}t$, and $\{x \in \mathbb{R},\ y \in (-\infty, x]\} = \{y \in \mathbb{R},\ x \in [y, +\infty)\} = \{t \in \mathbb{R},\ s \in [0, +\infty)\}$. We have thus obtained that $A_v = A_f$ as distributions, and we know that this means that $f = v$ almost everywhere (see **§13.5**), that is, (104). □

Exercise 13.19. Show the following proposition:

Proposition 13.20. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C(\Omega)$ a continuous function. Suppose that, for every $j \in \{1, \dots, n\}$, there is a continuous function $g_j \in C(\Omega)$ such that $\mathrm{D}^j A_f = A_{g_j}$, i.e., $\mathrm{D}^j f = g_j$ in distributional sense. Then $f \in C^1(\Omega)$ and $\mathrm{D}^j f = g_j$.*

◊

Exercise 13.21. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with bounded variation. For instance, the Cantor staircase function. Show that $\mathrm{D}A_f = A_\mu$, where $\mu \in \mathtt{Rad}(\mathbb{R})$ is the measure defined by

$$\mu([a, b)) = f(b) - f(a)$$

for all $a, b \in \mathbb{R}$ with $a < b$. For instance, if f is the Cantor staircase function, then we know that, for almost every $x \in \mathbb{R}$, f is differentiable at x and $f'(x) = 0$. However, $\mathrm{D}A_f \neq 0$.

Hint: see [11, §6.14]. ◊

§13.13. Intermezzo: Banach–Steinhaus Theorem. We will need to cite Banach–Steinhaus Theorem. Here we see a version of its statement that is less general than the original, but it is what we will need later on. Before stating the theorem, we fix the terms used.

A family Γ of linear functions $\gamma : X \to Y$ between topological vector spaces is *equicontinuous* if for every open neighborhood V of 0 in Y there exists an open neighborhood U of 0 in X such that $\gamma(U) \subset V$ for all $\gamma \in \Gamma$. As as short notation, for $\Gamma \subset \mathtt{Lin}(X; Y)$ and $E \subset X$, we define

$$\Gamma(E) = \{\gamma(x) : \gamma \in \Gamma,\ x \in E\} \subset Y.$$

In the finite-dimensional case, i.e., when both X and Y have finite dimension, equicontinuity is equivalent to boundedness (in the operator norm) and pre-compactness. In the infinite dimensional case, this is not the case.

A *Fréchet space* is a topological vector space (X, τ) such that

1. the topology τ is generated by a complete *invariant metric* (i.e., a complete distance function $d : X \times X \to (0, +\infty)$ such that $d(x + z, y + z) = d(x, y)$ for all $x, y, z \in X$);
2. X is locally convex, that is, for every $U \subset X$ open with $0 \in U$ there exists $V \subset U$ open with $0 \in V$ and V convex.

A subset $E \subset X$ in a topological vector space X is *bounded* if for every $U \subset X$ open with $0 \in U$ there exists $\lambda > 0$ such that $E \subset \lambda U$. This notion of boundedness might look abstract, but you can easily show the following statement: *If $E = \{x_j\}_{j\in\mathbb{N}}$ is a convergent sequence in X, then E is bounded.*

Exercise 13.22. Show that, if $E = \{x_j\}_{j\in\mathbb{N}}$ is a convergent sequence in X, then E is bounded. ♢

Theorem 13.23 ((Consequence of) Banach–Steinhaus Theorem)**.** *Suppose that X is a Fréchet space and Y a topological vector space, Γ is a collection of continuous linear maps from X to Y. If*

$$\forall x \in X \qquad \Gamma(x) \text{ is bounded in } Y,$$

then Γ is equicontinuous.

See [11, §2.1–§2.6], and also the Wikipedia page.

Corollary 13.24. *Let X be a Fréchet space, Y and Z topological vector spaces, and $B : X \times Y \to Z$ a bilinear map. Suppose that B is continuous in each entry separately, i.e., for every $x \in X$ the linear map $Y \to Z$, $y \mapsto B(x, y)$, is continuous, and for every $y \in Y$ the linear map $X \to Z$, $x \mapsto B(x, y)$, is continuous.*

If $\{x_j\}_{i\in\mathbb{N}} \subset X$ and $\{y_j\}_{j\in\mathbb{N}} \subset Y$ are sequences with $\lim_{j\to\infty} x_j = x_\infty$ in X and $\lim_{j\to\infty} y_j = y_\infty$ in Y, then

$$\lim_{j\to\infty} B(x_j, y_j) = B(x_\infty, y_\infty). \tag{105}$$

Proof. For $j \in \mathbb{N} \cup \{\infty\}$, define $b_j : X \to Z$, $b_j(x) = B(x, y_j)$. Set $\Gamma = \{b_j\}_{j\in\mathbb{N}\cup\{\infty\}}$. Since B is continuous in each entry, the functions b_j are continuous. If $x \in X$, since $\Gamma(x)$ is a sequence in Z convergent to $b_\infty(x)$, then $\Gamma(x)$ is bounded. By the Banach–Steinhaus Theorem 13.23, Γ is equicontinuous.

We are now ready to prove (105). Let $U \subset Z$ be a neighborhood of 0 in Z. We want to show that there is $N \in \mathbb{N}$ such that

$$\exists N \in \mathbb{N}\ \forall j > N \qquad B(x_j, y_j) \in B(x_\infty, y_\infty) + U. \tag{106}$$

As a general fact in topological vector spaces, there is $\tilde{U} \subset U$ neighborhood of 0 in Z such that $\tilde{U} - \tilde{U} \subset U$. Since Γ is equicontinuous, there is $V \subset X$ neighborhood of 0 in X such that $\Gamma(V) \subset \tilde{U}$. Since $x_n \to x_\infty$, then there is $N \in \mathbb{N}$ such that $x_n - x_\infty \in V$ for all $n > N$. Up to taking N larger, we have also $B(x_\infty, y_\infty - y_j) \in \tilde{U}$ for all $n > N$, because $y \mapsto B(x_\infty, y)$ is continuous and thus $\lim_{j\to\infty} B(x_\infty, y_\infty - y_j) = 0$. So, for $j > N$ we have

$$\begin{aligned} B(x_j, y_j) - B(x_\infty, y_\infty) &= B(x_j - x_\infty, y_j) - B(x_\infty, y_\infty - y_j) \\ &= b_j(x_j - x_\infty) - B(x_\infty, y_\infty - y_j) \in \tilde{U} - \tilde{U} \subset U. \end{aligned}$$

We have thus obtained (106). □

§13.14. Product of distributions. A derivative is usually defined using the Leibniz rule: $\partial(fg) = f\partial g + g\partial f$. The product of two distributions is not well defined... We will see special situations in which we can multiply two distributions, but there is not a general product of two distributions.

§13.15. Product of a smooth function and a distribution. If $A \in \mathscr{D}(\Omega)$ and $f \in C^\infty(\Omega)$, then we define fA by

$$(fA)[\phi] = A[f\phi].$$

Again, we justify this formula with Proposition 13.14: indeed, the map $\Phi_f : \phi \mapsto f\phi$ is a continuous linear operator $\mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$. So, $\Phi_f^* A = A \circ \Phi_f$ defines a continuous linear operator $\mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$.

Proposition 13.25. *If $f_k \to f_\infty$ in $C^\infty(\Omega)$ and $A_k \to A_\infty$ in $\mathscr{D}'(\Omega)$, then $f_k A_k \to f_\infty A_\infty$ in $\mathscr{D}'(\Omega)$.*

Proof. Consider the bilinear map $B : C^\infty(\Omega) \times \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$, $B(f, A) = fA$. We have seen that B is continuous in each entry separately. Since $C^\infty(\Omega)$ is a Fréchet space, we conclude thanks to Corollary 13.24 of the Banach–Steinhaus Theorem. □

Corollary 13.26. *If $\phi_j \to \phi_\infty$ in $\mathscr{D}(\Omega)$ and $A_j \to A$ in $\mathscr{D}'(\Omega)$, then* $\lim_{j\to\infty} A_j[\phi_j] = A[\phi_\infty]$.

Exercise 13.27. Prove Corollary 13.26 using Proposition 13.25. ♢

Exercise 13.28 (Generalized Leibniz Rule)**.** Show that, if $u \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, then, for every $\alpha \in \mathbb{N}^n$,

$$\mathrm{D}^\alpha(fu) = \sum_{\beta\leq\alpha} \binom{\alpha}{\beta} \mathrm{D}^\beta f \cdot \mathrm{D}^{\alpha-\beta} u. \tag{107}$$

Hint: First of all, understand this formula when u is a smooth function. Then consider the case $|\alpha| = 1$ (just one derivative). ♢

§13.16. Locality. We say that two distributions $A_1, A_2 \in \mathscr{D}'(\Omega)$ are equal on an open set $\omega \subset \Omega$, that is, $A_1 = A_2$ in ω, if $A_1\phi = A_2\phi$ for all $\phi \in \mathscr{D}(\omega)$.

Proposition 13.29. *Let $\mathscr{U}$ be an open cover of an open set $\Omega \subset \mathbb{R}^n$ and let $\{A_\omega\}_{\omega\in\mathscr{U}}$ be a collection of distributions with:*

(1) $A_\omega \in \mathscr{D}'(\omega)$ for all $\omega \in \mathscr{U}$, and,
(2) $A_{\omega_1} = A_{\omega_2}$ in $\omega_1 \cap \omega_2$ for all $\omega_1, \omega_2 \in \mathscr{U}$.

Then there exists a unique $A \in \mathscr{D}'(\Omega)$ with $A = A_\omega$ in ω for all $\omega \in \mathscr{U}$.

Proof. Let $\{\psi_j\}_{j\in\mathbb{N}}$ be a partition of unity subordinated to $\mathscr{U}$. More precisely:

(1) $\psi_j \in C_c^\infty(\Omega)$ for all j;
(2) for every $x \in \Omega$ the set $\{j : \psi_j(x) \neq 0\}$ is finite;
(3) $\sum_j \psi_j(x) = 1$ for all $x \in \Omega$;
(4) for every $j \in \mathbb{N}$ there is $\omega_j \in \mathscr{U}$ such that $\mathrm{spt}(\psi_j) \subset \omega_j$.

We fix the subcover $\{\omega_j\}_{j\in\mathbb{N}} \subset \mathscr{U}$.

Define $A : \mathscr{D}(\Omega) \to \mathbb{C}$ as follows: if $\phi \in \mathscr{D}(\Omega)$, then $A[\phi] = \sum_{j\in\mathbb{N}} A_{\omega_j}[\psi_j\phi]$. Notice that, since $\mathrm{spt}(\phi)$ is compact and the local finiteness of the cover $\{\omega_j\}_j$, the sum is a finite sum.

Clearly A is linear. If $\phi_k \to 0$ in $\mathscr{D}(\Omega)$, then there is $K \Subset \Omega$ with $\mathrm{spt}(\phi_k) \subset K$ for all k. So, there is $N \in \mathbb{N}$ with $K \subset \bigcup_{j=1}^N \omega_j$ and $A[\phi_k] = \sum_{j=1}^N A_{\omega_j}[\psi_j\phi_k]$ for all k. We conclude that $\lim_{k\to\infty} A[\phi_k] = \sum_{j=1}^N \lim_{k\to\infty} A_{\omega_j}[\psi_j \lim_{k\to\infty}\phi_k] = 0$.

By Proposition 13.5, we obtain that $A \in \mathscr{D}'(\Omega)$.

If $\omega \in \mathscr{U}$, then, for every $\phi \in \mathscr{D}(\omega)$, we have $A[\phi] = \sum_{j\in\mathbb{N}} A_{\omega_j}[\psi_j\phi] = \sum_{j\in\mathbb{N}} A_\omega[\psi_j\phi] = A_\omega[\sum_{j\in\mathbb{N}}\psi_j\phi] = A_\omega[\phi]$, where we used the fact that $\mathrm{spt}(\psi_j\phi) \subset \mathrm{spt}(\psi_j)\cap\mathrm{spt}(\phi) \subset \omega_j\cap\omega$, that $A_{\omega_j} = A_\omega$ on $\omega_j \cap \omega$, that there is a finite set $J \subset \mathbb{N}$ such that $\mathrm{spt}(\psi_j)\cap\mathrm{spt}(\phi) \neq \emptyset$, and that, for every $x \in \mathrm{spt}(\phi)$, $\sum_{j\in\mathbb{N}}\psi_j(x) = \sum_{j\in J}\psi_j(x) = 1$.

Finally, we need to show uniqueness. Suppose there is $\bar{A} \in \mathscr{D}'(\Omega)$ such that $\bar{A} = A_\omega$ on ω for every $\omega \in \mathscr{U}$. Then, for every $\phi \in \mathscr{D}(\Omega)$, $\bar{A}[\phi] = \bar{A}[\sum_j \psi_j\phi] = \sum_j \bar{A}[\psi_j\phi] = \sum_j A_{\omega_j}[\psi_j\phi] = A[\phi]$. Hence, $\bar{A} = A$. □

Exercise 13.30. Show that a partition of unity exists for every open cover of an open subset of $\mathbb{R}^n$. In other words, if $\Omega \subset \mathbb{R}^n$ is open and $\mathscr{U}$ is a collection of open subsets of Ω such that $\bigcup\mathscr{U} = \Omega$, then there is a countable family of functions $\{\psi_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega)$ such that

(1) $0 \leq \psi_j(x) \leq 1$ for all j and all $x \in \Omega$
(2) for every $x \in \Omega$ the set $\{j : \psi_j(x) \neq 0\}$ is finite;
(3) $\sum_j \psi_j(x) = 1$ for all $x \in \Omega$;
(4) for every $j \in \mathbb{N}$ there is $\omega_j \in \mathscr{U}$ such that $\mathrm{spt}(\psi_j) \subset \omega_j$.

Moreover, show that, if $K \subset \Omega$ is compact, then there is a finite set $J \subset \mathbb{N}$ such that $\sum_{j\in J}\psi_j(x) = 1$ for all $x \in K$.

Hint: Rudin has a proof. ♢

Exercise 13.31. Let $A \in \mathscr{D}'(\Omega)$ and $\mathscr{U}$ an open cover of Ω. Show that, if $\bar{A} \in \mathscr{D}'(\Omega)$ is such that $\bar{A} = A$ on ω, for every $\omega \in \mathscr{U}$, then $\bar{A} = A$.

The meaning of this exercise is as follows: If we start from some $A \in \mathscr{D}'(\Omega)$, then we can localize A to each open subset of some given cover of Ω, and then we can recover A from Proposition 13.29. $\Diamond$

Corollary 13.32. *If $A \in \mathscr{D}'(\Omega)$ is such that for every $x \in \Omega$ there is $\omega \subset \Omega$ open with $x \in \omega$ and $A = 0$ on ω, then $A = 0$ on Ω.*

Proof. Define $\mathscr{U}$ as the open cover of Ω given by the open sets $\omega \subset \Omega$ with $A = 0$ on ω. Proposition 13.29 claims that there exists a unique distribution in $\mathscr{D}'(\Omega)$ that is equal to A on each $\omega \in \mathscr{U}$. Since 0 is one of such distributions, uniqueness implies $A = 0$. $\square$

Remark 13.33. Corollary 13.32 might look confusionally trivial. To understand it better, it is useful to think of a situation (outside the world of distributions, of course) where locality fails. Here is one example.

Consider the space $\mathscr{C} = \{\alpha \in \Omega^1(\mathbb{R}^2 \setminus \{0\}) : \mathrm{d}\alpha = 0\}$ of closed 1-forms on the plane punctured plane. We consider (continuous) linear functionals $\mathscr{C} \to \mathbb{C}$. Let $\mathbb{S}^1 = \{z \in \mathbb{R}^2 : |z| = 1\}$ the the unit circle. Define $A[\alpha] := \int_{\mathbb{S}^1} \alpha = \int_0^{2\pi} \langle \alpha(e^{it}) | ie^{it} \rangle \,\mathrm{d}t$ (we identify $\mathbb{C}$ with $\mathbb{R}^2$ for notational purposes). Notice that A is not zero: for example, $A[x\mathrm{d}y - y\mathrm{d}x] \neq 0$.

Now, we claim that, for every $x \in \mathbb{R}^2$, there exists $\omega \subset \mathbb{R}^2$ open neighborhood of x such that $A[\alpha] = 0$ for all $\alpha \in \mathscr{C}$ with $\mathrm{spt}(\alpha) \subset \omega$. Indeed, if $x \notin \mathbb{S}^1$, then we can just take ω with $\omega \cap \mathbb{S}^1 = \emptyset$. If $x \in \mathbb{S}^1$, then we can take $\omega = B(x, 1/2)$, the ball of radius 1/2 and center x. If α is a closed 1 form on $\mathbb{R}^2 \setminus \{0\}$, the integral of α over each contractible loop is 0. So, if $\mathrm{spt}(\alpha) \subset \omega$, and if $y, z \in \mathbb{R}^2$ are the two extremal points of the arc $\mathbb{S}^1 \cap \omega$, then the integral of α over $\mathbb{S}^1 \cap \omega$ is equal to the integral of α along another path from y to z that does not intersect the support of α (such as the boundary of ω itself). It follows that the integral of α along $\mathbb{S}^1 \cap \omega$ is zero. This shows the claim.

§13.17. Support of a distribution. The support of $A \in \mathscr{D}'(\Omega)$ is defined as the complement of the set

$$\Omega \setminus \mathrm{spt}(A) = \bigcup \{U \subset \Omega \text{ open, such that } A\phi = 0 \ \forall \phi \in \mathscr{D}(\Omega)\}.$$

Proposition 13.34. *Let $A \in \mathscr{D}'(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$. If $\mathrm{spt}\phi \subset \Omega \setminus \mathrm{spt}A$, then $A\phi = 0$.*

Proof. Let $\mathscr{U} = \{\omega \subset \Omega : A = 0 \text{ in } \omega\}$. Then $\mathscr{U}$ is a cover of $\Omega \setminus \mathrm{spt}A$, by definition of support of a distribution. Let $\{\psi_j\}_{j \in \mathbb{N}}$ be a partition of unity subordinated to $\mathscr{U}$. Since $\mathrm{spt}\phi$ is a compact subset of $\Omega \setminus \mathrm{spt}A$, there is N such that $\phi = \sum_{j=1}^N \psi_j \phi$. So, $A\phi = \sum_{j=1}^N A[\psi_j \phi] = 0$.

We can give another proof using Proposition 13.29: Let $V = \Omega \setminus \mathrm{spt}A$. Then V is open and $\mathscr{U}$ is an open cover of V. By the very definition of $\mathscr{U}$, each restriction A_ω of A to $\omega \in \mathscr{U}$ is zero. By Proposition 13.29, these $A_\omega \in \mathscr{D}'(\omega)$ (which are zero) are the restrictions of a unique distribution on V. Since 0 is a distribution on V so that $0 = A_\omega$ on ω for each $\omega \in \mathscr{U}$, then $A = 0$ on V by uniqueness. $\square$

A few examples of support:

- If $A_f \in \mathscr{D}'(\Omega)$ is the distribution associated to $f \in C(\Omega)$, then $\mathrm{spt}(A_f) = \mathrm{spt}(f)$.
- $\mathrm{spt}(\delta_p) = \{p\}$.

Exercise 13.35. Show the following statement: if $A \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$ are such that $\mathrm{spt}A \subset \{f = 1\}$, then $fA = A$. $\Diamond$

Exercise 13.36. Show the following statement: if $A \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, then $\mathrm{spt}(fA) \subset \mathrm{spt}(f) \cap \mathrm{spt}(A)$. Is equality true?

Hint for question: Try with $A = \delta_0$. $\Diamond$

§13.18. Derivatives of the Dirac delta. I don't think we will ever use the following statement, but it is important to know it.

Proposition 13.37. *Suppose $A \in \mathscr{D}'(\Omega)$, $p \in \Omega$ and $\mathrm{spt}(A) = \{p\}$. Then there is $N \in \mathbb{N}$ and constants $c_\alpha \in \mathbb{C}$ such that*

$$A = \sum_{|\alpha| \le N} c_\alpha \mathrm{D}^\alpha \delta_p, \tag{108}$$

where $\delta_p\phi = \phi(p)$ is the Dirac delta, see **§13.6**.

Conversely, if A is as in (108)*, then* $\operatorname{spt}(A) = \{p\}$*, unless $c_\alpha = 0$ for all α.*

Proof. See [11, Thm.6.21]. □

§13.19. Distributions as derivatives of functions. I don't think we will ever use the following statement, but it is important to know it.

Theorem 13.38. *For ever $A \in \mathscr{D}'(\Omega)$ there are continuous functions $g_\alpha : \Omega \to \mathbb{C}$ for $\alpha \in \mathbb{N}^n$ such that the infinite sum $\sum_\alpha g_\alpha$ is locally finite and*

$$A = \sum_{\alpha\in\mathbb{N}^n} \mathrm{D}^\alpha g_\alpha.$$

Proof. See [11, Thm.6.28]. □

Exercise 13.39. Use Theorem 13.38 to show the following: if $A \in \mathscr{D}'(\mathbb{R})$, then there exists $g \in C(\mathbb{R})$ and $N \in \mathbb{N}$ such that $A = g^{(N)}$ (N-th distributional derivative of g). ♢

Remark 13.40. Here is an example of the situation described by Theorem 13.38. On $\mathbb{R}$, consider the Dirac delta at zero δ_0. The theorem claims that there is a continuous function $g : \mathbb{R} \to \mathbb{C}$ and N such that $g^{(N)} = \delta_0$, see also Exercise 13.39. In fact, if $g(x) = \int_{-\infty}^x \mathbb{1}_{[0,+\infty)}(y)\,\mathrm{d}y$, then (Exercise!) $g^{(2)} = \delta_0$.

§13.20. Convolution of functions. If $f, g : \mathbb{R}^n \to \mathbb{C}$ are measurable functions, we define

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y, \tag{109}$$

whenever the integral is well defined, i.e., whenever $y \mapsto f(y)g(x-y)$ is integrable. Notice that, as soon as $y \mapsto f(y)g(x-y)$ is integrable, then not only $f * g(x)$ is well defined, but also

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,\mathrm{d}y = \int_{\mathbb{R}^n} f(x-y)g(y)\,\mathrm{d}y = g * f(x).$$

Exercise 13.41. Show that, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then $f * g(x)$ is well defined for all $x \in \mathbb{R}^n$ and in fact $f * g \in C^0_b(\mathbb{R}^n)$. ♢

Exercise 13.42 (Young's inequality)**.** Show that, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $p \in [1,\infty]$, then $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p} \tag{110}$$

Hint. By Hölder inequality, with $\frac{1}{p} + \frac{1}{p'} = 1$, $\int |f(y)g(y-x)|\,\mathrm{d}y = \int |f(y)|^{1/p'} \cdot |f(y)|^{1/p}|g(y-x)|\,\mathrm{d}y \le (\int |f(y)|\,\mathrm{d}y)^{1/p'} \cdot (\int |f(y)||g(y-x)|^p\,\mathrm{d}y)^{1/p}$. Therefore, $\int (f * g(x))^p\,\mathrm{d}x \le (\int |f(y)|\,\mathrm{d}y)^{p/p'} \cdot \int\int |f(y)||g(y-x)|^p\,\mathrm{d}y\,\mathrm{d}x \le (\int |f(y)|\,\mathrm{d}y)^{p/p'} \cdot \int |g(y)|^p\,\mathrm{d}y \cdot \int |f(y)|\,\mathrm{d}y$. ♢

Exercise 13.43. Show that, if $f, g \in C^0(\mathbb{R}^n)$, then

$$\operatorname{spt}(f * g) \subset \operatorname{spt}(f) + \operatorname{spt}(g).$$

Can you find a case where equality holds? And where equality does not hold? ♢

§13.21. Translations and inversion of a distribution. We will extend the definition of convolution to distributions. To do so, we need two linear operators on distributions: here they are.

For $x \in \mathbb{R}^n$, define $\tau_x : \mathbb{R}^n \to \mathbb{R}^n$ by $\tau_x(y) = y - x$. For $x \in \mathbb{R}^n$, define the continuous linear operator $\tau_x : \mathscr{D} \to \mathscr{D}$ by

$$\tau_x\phi = \phi \circ \tau_x,$$

for all $\phi \in \mathscr{D}$. In other words, $\tau_x\phi = \tau_x^*\phi$ is the pull-back via τ_x. By Proposition 13.14, we have a continuous linear operator $\tau_x : \mathscr{D}' \to \mathscr{D}'$

$$\tau_x A[\phi] = A[\phi \circ \tau_x].$$

In other words, $\tau_x = (\tau_x^*)_*$ is the push forward induced by the pull back induced by τ_x.

$$\mathbb{R}^n \xrightarrow[y\mapsto y-x]{\tau_x} \mathbb{R}^n$$

$$\mathscr{D} \xleftarrow[\phi\mapsto\phi\circ\tau_x]{\tau_x^*} \mathscr{D}$$

$$\mathscr{D}' \xrightarrow[A\mapsto(\phi\mapsto A[\phi\circ\tau_x])]{(\tau_x^*)_*} \mathscr{D}'$$

I think at this point the amount of confusion surpass the amount of information.

Next, define the continuous linear operator $\mathscr{D}\to\mathscr{D}$, $\phi\mapsto\check{\phi}$, by

$$\check{\phi} : x \mapsto \phi(-x).$$

The notation $\check{\phi}$ will clash with the inverse of the Fourier transform: I do not know how to avoid this clash at the moment. For now, we follow Rudin's notation.

Exercise 13.44. Show the relations

$$\tau_y\tau_z = \tau_{y+z};$$

$$(\tau_x\phi)^\vee = \tau_{-x}\check{\phi}; \tag{111}$$

$$\tau_x(\mathrm{D}^\alpha\phi)^\vee = (-1)^{|\alpha|}\mathrm{D}^\alpha(\tau_x\check{\phi}). \tag{112}$$

$\Diamond$

§13.22. Convolution of a distribution. Let $\phi \in C_c^\infty(\mathbb{R}^n) = \mathscr{D}$ and $u \in \mathscr{D}'$. Define $\phi * u : \mathbb{R}^n \to \mathbb{C}$ by

$$(u*\phi)(x) = u[\tau_x\check{\phi}]. \tag{113}$$

Note that $\tau_x\check{\phi}(y) = \check{\phi}(y-x) = \phi(x-y)$. Notice that, if u is a function, then

$$(u*\phi)(x) \overset{(113)}{=} u[\tau_x\check{\phi}] = \int u(y)(\tau_x\check{\phi})(y)\,\mathrm{d}y = \int u(y)\phi(-(y-x))\,\mathrm{d}y \overset{(109)}{=} u*\phi(x).$$

Exercise 13.45. Show that, if $u \in \mathscr{D}'$ and $\phi\in\mathscr{D}$, then

$$u[\phi] = (u*\check{\phi})(0). \tag{114}$$

$\Diamond$

Exercise 13.46. Show that, if $u \in \mathscr{D}'$ and $\phi\in\mathscr{D}$, then

$$\mathrm{spt}(u*\phi) \subset \mathrm{spt}(u) + \mathrm{spt}(\phi) = \{x+y : x\in\mathrm{spt}(u),\ y\in\mathrm{spt}(\phi)\}. \tag{115}$$

Solution. Notice that, if $x\in\mathbb{R}^n$ and $\phi:\mathbb{R}^n\to\mathbb{C}$, then $\tau_x\check{\phi}(y) = \phi(x-y) \neq 0$ if and only if $x-y\in\mathrm{spt}(\phi)$, if and only if $y \in x-\mathrm{spt}(\phi)$. Therefore, $\mathrm{spt}(\tau_x\check{\phi}) = x-\mathrm{spt}(\phi)$.

Let $x\in\mathbb{R}^n\setminus(\mathrm{spt}(u)+\mathrm{spt}(\phi))$. Notice that, if $y\in\mathrm{spt}(\tau_x\check{\phi})\cap\mathrm{spt}(u)$, then there is $z\in\mathrm{spt}(\phi)$ with $y = x-z\in\mathrm{spt}(u)$, thus $x = y+z\in\mathrm{spt}(u)+\mathrm{spt}(\phi)$. Since $x\notin\mathrm{spt}(u)+\mathrm{spt}(\phi)$, then $\mathrm{spt}(\tau_x\check{\phi})\cap\mathrm{spt}(u)=\emptyset$. We conclude that $u*\phi(x) = u[\tau_x\check{\phi}] = 0$. $\Diamond$

Exercise 13.47. Show that, if $u\in\mathscr{D}'$, $\phi\in\mathscr{D}$ and $v\in\mathbb{R}^n$, then

$$u*(\tau_v\phi) = \tau_v(u*\phi). \tag{116}$$

$\Diamond$

Exercise 13.48. Show that $\phi\mapsto u*\phi$ is linear. $\Diamond$

Exercise 13.49. Show that $\delta_0*\phi = \phi$ for every $\phi\in\mathscr{D}$. What is $\delta_v*\phi$? $\Diamond$

Proposition 13.50. *If $u\in\mathscr{D}'$ and $\phi\in\mathscr{D}$, then $u*\phi\in C^\infty(\mathbb{R}^n)$ and, for every $\alpha\in\mathbb{N}^n$,*

$$\mathrm{D}^\alpha(u*\phi) = u*(\mathrm{D}^\alpha\phi) = (\mathrm{D}^\alpha u)*\phi. \tag{117}$$

Proof. If $v \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, then

$$\begin{aligned} \mathrm{D}_v(u * \phi)(x) &= \lim_{h\to 0} \frac{u * \phi(x+hv) - u * \phi(x)}{h} \\ &= \lim_{h\to 0} \frac{u[\tau_{x+hv}\check{\phi}] - u[\tau_x\check{\phi}]}{h} \\ &= \lim_{h\to 0} u_y \left[\frac{\phi(x+hv-y) - \phi(x-y)}{h} \right]. \end{aligned}$$

Since $\phi(x+hv-y) - \phi(x-y) \to \mathrm{D}_v\phi(x-y)$ in $\mathscr{D}_y$ as $h \to 0$, then

$$\lim_{h\to 0} u_y \left[\frac{\phi(x+hv-y) - \phi(x-y)}{h} \right] = u_y[\mathrm{D}_v\phi(x-y)] = u * \mathrm{D}_v\phi(x).$$

This shows that, for every $x \in \mathbb{R}^n$,

$$\mathrm{D}^j(u * \phi)(x) = (u * \mathrm{D}^j\phi)(x).$$

Iterating, we get the first part of (117).

Next,

$$\begin{aligned} u * (\mathrm{D}^\alpha\phi)(x) &= u[\tau_x(\mathrm{D}^\alpha\phi)^\vee] \\ &\overset{(112)}{=} (-1)^{|\alpha|} u[\mathrm{D}^\alpha(\tau_x\check{\phi})] \\ &= \mathrm{D}^\alpha u[\tau_x\check{\phi}] \\ &= ((\mathrm{D}^\alpha u) * \phi)(x). \end{aligned}$$

We thus have completed the proof of (117). □

Proposition 13.51. *If $u \in \mathscr{D}'$ and $\phi, \psi \in \mathscr{D}$, then*

$$u * (\phi * \psi) = (u * \phi) * \psi. \tag{118}$$

Proof. We use Lemma 13.52 and Proposition 13.50:

$$\begin{aligned} u * (\phi * \psi)(x) &= u[\tau_x(\phi * \psi)^\vee] \\ &= u_y[(\phi * \psi)^\vee(y-x)] \\ &= u_y[(\phi * \psi)(x-y)] \\ &= u_y \left[\int_{\mathbb{R}^n} \phi(x-y-z)\psi(z)\,\mathrm{d}z \right] \\ &= u_y \left[\lim_{h\to 0} \sum_{z\in\mathbb{Z}^n} h^n \phi(x-y-hz)\psi(hz) \right] \\ \text{[by Lemma 13.52]} &= \lim_{h\to 0} u_y \left[\sum_{z\in\mathbb{Z}^n} h^n \phi(x-y-hz)\psi(hz) \right] \\ \text{[since the sum is finite]} &= \lim_{h\to 0} \sum_{z\in\mathbb{Z}^n} h^n u_y[\phi(x-y-hz)]\psi(hz) \\ [\phi(x-y-hz) = \tau_{x-hz}\check{\phi}(y)] &= \lim_{h\to 0} \sum_{z\in\mathbb{Z}^n} h^n (u * \phi)(x-hz)]\psi(hz) \\ \text{[by Lemma 13.52 and Proposition 13.50]} &= \int_{\mathbb{R}^n} (u * \phi)(x-z)\psi(z)\,\mathrm{d}z \\ &= (u * \phi) * \psi(x). \end{aligned}$$

□

Lemma 13.52. *Let $\phi \in \mathscr{D}$ and $\psi \in \mathscr{E}$. For $h > 0$, define*

$$\rho_h(x) = \sum_{z\in\mathbb{Z}^n} h^n \phi(x-hz)\psi(hz).$$

Then $\rho_h \in \mathscr{E}$ for all $h > 0$ and $\lim_{h\to 0} \rho_h = \phi * \psi$ in $\mathscr{E}$.

Moreover, if $\psi \in \mathscr{D}$, then $\lim_{h\to 0} \rho_h = \phi * \psi$ in $\mathscr{D}$.

Proof. Clearly ρ_h is smooth. The support of ρ_h is contained in $\mathrm{spt}(\phi)+\mathrm{spt}(\psi)$ (cfr. Exercise 13.43). Since the function $(x,y)\mapsto\phi(x-y)\psi(y)$ is continuous, then $\rho_h\to\phi*\psi$ uniformly on compact sets (see Exercise 13.55). The derivatives of ρ_h have the same form, i.e.

$$\mathrm{D}^\alpha\rho_h(x)=\sum_{z\in\mathbb{Z}^n}h^n\mathrm{D}^\alpha_x\phi(x-hz)\psi(hz).$$

So, $\mathrm{D}^\alpha\rho_h\to(\mathrm{D}^\alpha\phi)*\psi=\mathrm{D}^\alpha(\phi*\psi)$ uniformly. We obtain $\rho_h\to\phi*\psi$ in $\mathscr{E}$.

If $\psi\in\mathscr{D}$, then $(x,y)\mapsto\phi(x-y)\psi(y)$ is uniformly continuous with compact support. Reasoning as above, we obtain $\rho_h\to\phi*\psi$ in $\mathscr{D}$. □

Exercise 13.53. In this exercise, you show that Riemann sums converge to the integral.

Let $f:\mathbb{R}^n\to\mathbb{C}$ be a continuous and integrable function. (Integrable: $\int_{\mathbb{R}^n}|f(z)|\,\mathrm{d}z<\infty$). For $h>0$, define

$$F_h=\sum_{z\in\mathbb{Z}^n}h^nf(hz).$$

Show that $\lim_{h\to0}F_h=\int_{\mathbb{R}^n}f(z)\,\mathrm{d}z$. ♢

Exercise 13.54. [To do while listening to Paganini's Caprice No. 24]. Variation over Exercise 13.53: Let $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ be a uniformly continuous and integrable function. Define $F:\mathbb{R}^n\to\mathbb{C}$ by

$$F(x)=\int_{\mathbb{R}^n}f(x,z)\,\mathrm{d}z.$$

For $h>0$ and $x\in\mathbb{R}^n$, define

$$F_h(x)=\sum_{z\in\mathbb{Z}^n}h^nf(x,hz).$$

Show that $F_h\to F$ uniformly in x as $h\to0$. ♢

Exercise 13.55. Variation over Exercise 13.54: Let $f:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{C}$ be a ~~uniformly~~ continuous and integrable function. Define $F:\mathbb{R}^n\to\mathbb{C}$ by

$$F(x)=\int_{\mathbb{R}^n}f(x,z)\,\mathrm{d}z.$$

For $h>0$ and $x\in\mathbb{R}^n$, define

$$F_h(x)=\sum_{z\in\mathbb{Z}^n}h^nf(x,hz).$$

Show that $F_h\to F$ uniformly on compact sets in x as $h\to0$. ♢

§13.23. Smooth approximation of a distribution. Let $\rho\in C^\infty_c(\mathbb{R}^n)$ be such that $\mathrm{spt}(\rho)=B(0,1)$, $0\le\rho\le1$, $\check\rho=\rho$, and $\int\rho\,\mathrm{d}x=1$. Define $\rho_\epsilon(x)=\rho(x/\epsilon)/\epsilon^n$. We call the family $\{\rho_\epsilon\}_{\epsilon>0}$ an *approximation of the identity on* $\mathbb{R}^n$, or *a family of mollifiers*. For example, one can take

$$\rho(x)=\begin{cases}k\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1\\ 0 & \text{otherwise,}\end{cases}$$

where k normalizes the integral.

Exercise 13.56. Show that the function

$$\rho(x)=\begin{cases}\exp\left(\frac{1}{|x|^2-1}\right) & \text{if } |x|<1\\ 0 & \text{otherwise,}\end{cases}$$

is C^∞-smooth on $\mathbb{R}^n$ and compute $\int_{\mathbb{R}}^n\rho(x)\,\mathrm{d}x$. Show also that ρ is not analytic. Does it exist a family of analytic mollifiers?

Finally, why have I put the square in the definition of ρ? Do you think we could do without?

Hint: I have put the square to help you. The square is itself a hint. ♢

Proposition 13.57. *Let $\{\rho_\epsilon\}_{\epsilon>0}$ be an approximation of the identity on $\mathbb{R}^n$, $\phi \in \mathscr{D}$ and $u \in \mathscr{D}'$. Then*

$$\begin{aligned} \lim_{\epsilon\to 0} \phi * \rho_\epsilon &= \phi \text{ in } \mathscr{D}, \\ \lim_{\epsilon\to 0} u * \rho_\epsilon &= u \text{ in } \mathscr{D}'. \end{aligned} \tag{119}$$

Proof. The first identity was an exercise, but here is my solution. Notice that $\mathrm{spt}(\rho_\epsilon) = B(0,\epsilon)$. Since $\phi \in C^1_c$, then there is L such that $|\phi(x) - \phi(y)| \le L|x-y|$ for all x, y (see Exercise 13.58). Then, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} |\phi * \rho_\epsilon(x) - \phi(x)| &= \left| \int \phi(y)\rho_\epsilon(x-y)\,\mathrm{d}y - \phi(x)\int \rho_\epsilon(x-y)\,\mathrm{d}y \right| \\ &\le \int |\phi(y) - \phi(x)|\rho_\epsilon(x-y)\,\mathrm{d}y \\ &\le L\epsilon \int \rho_\epsilon(x-y)\,\mathrm{d}y = L\epsilon. \end{aligned}$$

This shows that $\phi * \rho_\epsilon \to \phi$ uniformly. Since this holds for every $\phi \in \mathscr{D}$, we also have that $\mathrm{D}^\alpha(\phi * \rho_\epsilon) = \phi * \mathrm{D}^\alpha \rho_\epsilon \to \mathrm{D}^\alpha \phi$ uniformly, for every $\alpha \in \mathbb{N}^n$. Therefore, $\phi * \rho_\epsilon \to \phi$ in $\mathscr{D}$.

For the second identity,

$$\begin{aligned} \lim_{\epsilon\to 0} u * \rho_\epsilon[\phi] &\overset{(114)}{=} \lim_{\epsilon\to 0} ((u * \rho_\epsilon) * \check{\phi})(0) \\ &\overset{(118)}{=} \lim_{\epsilon\to 0} (u * (\rho_\epsilon * \check{\phi}))(0) \\ [\text{because } \check{\rho} = \rho] &= \lim_{\epsilon\to 0} (u * (\check{\rho}_\epsilon * \check{\phi}))(0) \\ [\text{because } \check{f} * \check{g} = (f*g)^\vee] &= \lim_{\epsilon\to 0} (u * (\rho_\epsilon * \phi)^\vee)(0) \\ &\overset{(114)}{=} \lim_{\epsilon\to 0} u[\rho_\epsilon * \phi] \\ &\overset{(119)}{=} u[\phi]. \end{aligned}$$

□

Exercise 13.58. Let $\Omega \subset \mathbb{R}^n$ convex and $\phi \in C^1(\Omega)$ such that $L = \|\nabla\phi\|_{L^\infty} < \infty$ Show that, for every $x, y \in \Omega$, $|\phi(x) - \phi(y)| \le L|x-y|$.

Question: what happens if we drop the hypothesis of Ω being convex? ◊

Corollary 13.59. *The space $C^\infty(\mathbb{R}^n)$ is dense in $\mathscr{D}'$ (with respect to the topology of $\mathscr{D}'$).*

Exercise 13.60. Show that, if $\Omega \subset \mathbb{R}^n$ is open, then the space $C^\infty(\Omega)$ is dense in $\mathscr{D}'(\Omega)$ (with respect to the topology of $\mathscr{D}'(\Omega)$). ◊

Exercise 13.61. Is $\mathscr{D}(\Omega)$ dense in $\mathscr{D}'(\Omega)$? (Try at least for $\Omega = \mathbb{R}^n$).

Hint: Take $A[\phi] = \int \phi\,\mathrm{d}x$ and try to approximate A with functions in $C^\infty_c(\Omega)$. ◊

§13.24. Constancy theorem.

Theorem 13.62 (Constancy theorem)**.** *If $u \in \mathscr{D}'(\Omega)$ is such that $\partial_j u = 0$ for all $j \in \{1, \dots, n\}$, then u is a constant function, that is, there is $c \in \mathbb{C}$ such that $u[\phi] = c\int_\Omega \phi(x)\,\mathrm{d}x$ for all $\phi \in \mathscr{D}(\Omega)$.*

Proof. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be an approximation of the identity on $\mathbb{R}^n$. Then $\partial_j(u * \rho_\epsilon) = (\partial_j u) * \rho_\epsilon = 0$, therefore $u * \rho_\epsilon = c_\epsilon \in \mathbb{C}$. Moreover, by Proposition 13.57, $u[\phi] = \lim_{\epsilon\to 0} u * \rho_\epsilon[\phi] = \lim_{\epsilon\to 0} c_\epsilon \int \phi(y)\,\mathrm{d}y$. Therefore, the is $c \in \mathbb{C}$ with $c = \lim_{\epsilon\to 0} c_\epsilon$ and $u[\phi] = c\int \phi(y)\,\mathrm{d}y$, that is, u is a constant. □

§13.25. The space of all smooth functions as a Fréchet space. We define $\mathscr{E}(\Omega)$ as the vector space $C^\infty(\Omega)$ of smooth functions $\Omega \to \mathbb{C}$, endowed with the family of quasinorms $p_{N,K} : \mathscr{E}(\Omega) \to [0, +\infty)$, for $K \Subset \Omega$ and $N \in \mathbb{N}$, where

$$p_{N,K}(f) = \sup\{|\mathrm{D}^\alpha f(x)| : x \in K,\ |\alpha| \le N\} = \|f\|_{C^N(K)}.$$

The quasinorms determine a topology on $\mathscr{E}(\Omega)$, where $f_j \to f_\infty$ in

$$f_j \to f_\infty \text{ in } \mathscr{E}(\Omega) \quad \Leftrightarrow \quad \begin{array}{c} \forall K \Subset \Omega,\ \forall N \in \mathbb{N} \\ \lim_{j\to\infty} p_{N,K}(f_j - f_\infty) = 0. \end{array}$$

The topological vector space $\mathscr{E}(\Omega)$ is a so-called *Fréchet space.*

Exercise 13.63. Check whether the topology on $\mathscr{E}(\Omega)$ is the the initial topology induced by the maps $C^\infty(\Omega) \hookrightarrow C^m(\Omega)$.

Hint: I actually don't know if this is true. So, please send me an email with the result, if you don't mind. ♢

Notice that $\mathscr{D}(\Omega) \subset \mathscr{E}(\Omega)$ as sets, but also $\mathscr{D}(\Omega) \hookrightarrow \mathscr{E}(\Omega)$ continuously. Indeed, if $\phi_j \to 0$ in $\mathscr{D}(\Omega)$, then $\phi_j \to 0$ in $\mathscr{E}(\Omega)$, and thus the immersion is continuous by Proposition 13.5.

However, $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$ with the topology of $\mathscr{E}(\Omega)$.

Exercise 13.64. Show that $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$. ♢

Exercise 13.65. Find a sequence $\phi_j \in \mathscr{D}(\Omega)$ that converges to some f in $\mathscr{E}(\Omega)$ but it does not converge in $\mathscr{D}(\Omega)$.

Just as a note: there is a notion of "Cauchy sequence" for topological vector spaces (see Rudin's book). With such a notions available, one could check that $\mathscr{D}(\Omega)$ is complete in its own topology, but that its completion in the topology of $\mathscr{E}(\Omega)$ is $\mathscr{E}(\Omega)$. This should clarify the situation. ♢

§13.26. The dual space of $\mathscr{E}(\Omega)$... Let $\mathscr{E}'(\Omega)$ be the topological dual of $\mathscr{E}(\Omega)$, that is, the space of continuous linear functionals $\mathscr{E}(\Omega) \to \mathbb{C}$. A linear functional $A : \mathscr{E}(\Omega) \to \mathbb{C}$ is continuous if and only if, whenever $f_j \to f_\infty$ in $\mathscr{E}(\Omega)$, then $A[f_j] \to A[f_\infty]$ in $\mathbb{C}$.

The topology on $\mathscr{E}'(\Omega)$ is the usual weak* topology, that is, point-wise convergence:

$$E_j \to E_\infty \text{ in } \mathscr{E}'(\Omega) \quad \Leftrightarrow \quad \forall f \in \mathscr{E}(\Omega)\ \lim_{j\to\infty} E_j[f] = E_\infty[f].$$

§13.27. ...is made of distributions... Since $\mathscr{D}(\Omega) \subset \mathscr{E}(\Omega)$, then each $E \in \mathscr{E}(\Omega)$ is also a linear functional $E : \mathscr{D}(\Omega) \to \mathbb{C}$. Since $\mathscr{D}(\Omega) \hookrightarrow \mathscr{E}(\Omega)$ is continuous, or, otherwise said, since $\phi_j \to \phi_\infty$ in $\mathscr{D}(\Omega)$ implies $\phi_j \to \phi_\infty$ in $\mathscr{E}(\Omega)$, then also the restriction of E to $\mathscr{D}(\Omega)$ is continuous. In other words, $E \in \mathscr{D}(\Omega)$.

Moreover, if $E_j \to E_\infty$ in $\mathscr{E}'(\Omega)$, then clearly $E_j \to E_\infty$ in $\mathscr{D}'(\Omega)$. So, we can say $\mathscr{E}'(\Omega) \subset \mathscr{D}'(\Omega)$.

§13.28. ...with compact support. It remains to characterize the distributions $A \in \mathscr{D}'(\Omega)$ that belong to $\mathscr{E}'(\Omega)$:

Proposition 13.66.

$$\mathscr{E}'(\Omega) = \{A \in \mathscr{D}'(\Omega) \textit{ with compact support}\}.$$

Proof. $\boxed{\subset}$ From **§13.27**, we know that $\mathscr{E}'(\Omega) \subset \mathscr{D}'(\Omega)$. We need to show that, if $A \in \mathscr{E}'(\Omega)$, then $\mathrm{spt}(A)$ is compact. Arguing by contradiction, suppose this is not true, that is, that there is $A \in \mathscr{E}'(\Omega)$ with $\mathrm{spt}(A)$ not compact. Let $\{K_j\}_{j\in\mathbb{N}}$ be a countable increasing sequence of compact sets $K_j \subset \Omega$ such that $\Omega = \bigcup_{j\in\mathbb{N}} K_j$. Since $\mathrm{spt}(A)$ is not compact, then, for every $j \in \mathbb{N}$ there is $\phi_j \in \mathscr{D}(\Omega)$ with $\mathrm{spt}(\phi_j) \cap K_j = \emptyset$ and $A[\phi_j] = 1$.

Notice that $\phi_j \to 0$ in $\mathscr{E}(\Omega)$: indeed, if $K \subset \Omega$ is compact, then there is $k \in \mathbb{N}$ such that $K \subset K_k$ and thus $\phi_j|_K = 0$ for all $j > k$; hence, for every $K \subset \Omega$ compact and every $N \in \mathbb{N}$, $\lim_{j\to\infty} p_{N,K}(\phi_j) = 0$.

But we have assumed that $A \in \mathscr{E}'(\Omega)$, and therefore we should have $\lim_{j\to\infty} A[\phi_j] = 0$, in contradiction with $A[\phi_j] = 1$ for all j. We conclude that A must have compact support.

$\boxed{\supset}$ This inclusion does not make sense when taken literally: if $A \in \mathscr{D}'(\Omega)$, then A is a linear map $\mathscr{D}(\Omega) \to \mathbb{C}$, while $\mathscr{E}(\Omega)$ is a larger space than $\mathscr{D}(\Omega)$. So, A is NOT a function

$\mathscr{E}(\Omega) \to \mathbb{C}$, taken as it is. However, we need to consider that $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$: so, if there is a continuous linear functional $E : \mathscr{E}(\Omega) \to \mathbb{C}$ whose restriction to $\mathscr{D}(\Omega)$ is A, then E is uniquely determined by A.

This is to say that the correct interpretation of the inclusion $\boxed{\supset}$ is: *every $A \in \mathscr{D}'(\Omega)$ with compact support extends uniquely to a continuous linear functional $\bar{A} : \mathscr{E}(\Omega) \to \mathbb{C}$.*

Let $A \in \mathscr{D}'(\Omega)$ be with compact support. Fix some $\psi \in C_c^\infty(\Omega)$ such that $\mathrm{spt}A \subset \mathrm{interior}\{\psi = 1\}$, which exists because $\mathrm{spt}(A)$ is compact. Define $A_\psi : \mathscr{E}(\Omega) \to \mathbb{C}$ by $A_\psi f = A[\psi f]$. Notice that, firstly, $A_\psi \in \mathscr{E}'(\Omega)$: indeed, if $f_j \to f_\infty$ in $\mathscr{E}(\Omega)$, then $\psi f_j \to \psi f_\infty$ in $\mathscr{D}(\Omega)$, and thus $A_\psi[f_j] \to A_\psi[f_\infty]$ in $\mathbb{C}$.

Moreover, if $\phi \in \mathscr{D}(\Omega)$, then $\mathrm{spt}((1-\psi)\phi) \cap \mathrm{spt}(A) = \emptyset$ and thus $A[\phi] = A[\psi\phi + (1-\psi)\phi] = A[\psi\phi] = A_\psi[\phi]$. We conclude that the restriction of A_ψ to $\mathscr{E}(\Omega)$ is A. □

Exercise 13.67. In the proof of Proposition 13.66, we have defined the extension of $A \in \mathscr{D}'(\Omega)$ to $\mathscr{E}(\Omega)$ as $A_\psi[f] = A[\psi f]$, where $\psi \in C_c^\infty(\Omega)$ such that $\mathrm{spt}A \subset \mathrm{interior}\{\psi = 1\}$. It looks like this extension depends on the choice of ψ. Does it? ♢

Exercise 13.68. Show that all distributions in $\mathscr{E}'(\Omega)$ have finite order. More precisely, if $u \in \mathscr{E}'(\Omega)$, then there are $N \in \mathbb{N}$ and $C \in \mathbb{R}$ such that, for every $f \in \mathscr{E}(\Omega)$,

$$|u[f]| \le C\|f\|_{C^N(\mathrm{spt}(u))}.$$

♢

§13.29. Convolution of distributions with compact support. We have started with the convolution $u*v$ of functions u and v. We have extended this operation to convolution $u * v$ where $u \in \mathscr{D}'$ is a distribution and $v \in \mathscr{D}$ is a smooth function with compact support. As it happens with the product, convolution is not defined for arbitrary pairs of distributions. However, we can still push our definition of convolution $u * v$ to the case when $u, v \in \mathscr{D}'$ and *at least one of the distributions has compact support.* This is done in three steps.

$\boxed{\text{Step 1:}}$ If $u \in \mathscr{E}'$ and $f \in \mathscr{E}$, then we define

$$u * f(x) = u[\tau_x \check{f}], \tag{120}$$

exactly as we did for distributions in $\mathscr{D}'$. So, if $u \in \mathscr{D}'$ has compact support (i.e., $u \in \mathscr{E}'$), then $u * \phi$ is defined not only for $\phi \in \mathscr{D} = C_c^\infty(\mathbb{R}^n)$, but also for $\phi \in \mathscr{E} = C^\infty(\mathbb{R}^n)$.

Proposition 13.69. *If $u \in \mathscr{E}'$ and $f \in \mathscr{E}$, then $u * f \in \mathscr{E}$. Moreover*

$$\begin{aligned} &\mathrm{spt}(u * f) \subset \mathrm{spt}(u) + \mathrm{spt}(f), \ \textit{and} \\ &\mathrm{D}^\alpha(u * f) = u * (\mathrm{D}^\alpha f) = (\mathrm{D}^\alpha u) * f, \qquad \forall \alpha \in \mathbb{N}^n. \end{aligned}$$

In particular, if $f \in \mathscr{D}$, then $u * f \in \mathscr{D}$.

Proof. The proof is the same as for Proposition 13.50. In fact, I wonder if there is a way to merge the two proofs into one. □

$\boxed{\text{Step 2:}}$ We characterize continuous linear operators $\mathscr{D} \to \mathscr{E}$ that commute with translations. First of all, notice that, if $u \in \mathscr{D}'$, then $L : \phi \mapsto u * \phi$ is a linear function $\mathscr{D} \to \mathscr{E}$. Moreover, by (116), L commutes with translation, i.e., $L \circ \tau_v = \tau_v \circ L$ for all $v \in \mathbb{R}^n$. The next Lemma 13.70 shows that L is in fact continuous. Theorem 13.74 will finally prove that all continuous linear operators $\mathscr{D} \to \mathscr{E}$ that commute with translations are of this form.

Lemma 13.70. *If $\phi_j \to \phi_\infty$ in $\mathscr{D}$ and $u \in \mathscr{D}'$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{E}$.*

Proof. Let $\phi_j \to \phi_\infty$ in $\mathscr{D}$. We need to show that, if $\phi_j \to \phi_\infty$ in $\mathscr{D}$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{E}$, that is, $\mathrm{D}^\alpha(u * \phi_j) \to \mathrm{D}^\alpha(u * \phi_\infty)$ uniformly on compact sets, for every $\alpha \in \mathbb{N}^n$. In fact, since $\mathrm{D}^\alpha(u * \phi_j) = u * (\mathrm{D}^\alpha \phi_j)$ by (117), we only need to show that, if $\phi_j \to \phi_\infty$ in $\mathscr{D}$, then $u * \phi_j \to u * \phi_\infty$ uniformly on compact sets.

Let $\phi_j \to \phi_\infty$ in $\mathscr{D}$. Fix a compact set $K \subset \mathbb{R}^n$. Let $\tilde{K} \subset \mathbb{R}^n$ be a compact such that $\mathrm{spt}(\phi_j) \subset \tilde{K}$ for all j. Notice that, if $x \in K$, then $x - y \in \tilde{K}$ if and only if $y \in K - \tilde{K} = \{a - b : a \in K,\ b \in \tilde{K}\}$. So, for every $x \in K$ and $\phi \in \mathscr{D}$ with $\mathrm{spt}(\phi) \subset \tilde{K}$, then $\tau_x \check{\phi}$ is supported in $K - \tilde{K}$, which is compact.

By Proposition 13.9, there are $C \in \mathbb{R}$ and $N \in \mathbb{N}$ such that $|u[\phi]| \le C\|\phi\|_{C^N}$ for all $\phi \in \mathscr{D}$ with $\mathrm{spt}(\phi) \subset K - \tilde{K}$. Then, for every $x \in K$,

$$\begin{aligned}|u * \phi_j(x) - u * \phi_\infty(x)| &= |u[\tau_x(\check{\phi}_j)] - u[\tau_x(\check{\phi}_\infty)]| \\ &\le C\|\tau_x(\check{\phi}_j) - \tau_x(\check{\phi}_\infty)\|_{C^N} = C\|\phi_j - \phi_\infty\|_{C^n}.\end{aligned}$$

Therefore, $u * \phi_j \to u * \phi_\infty$ uniformly on K. □

Lemma 13.71. *If $\phi_j \to \phi_\infty$ in $\mathscr{E}$ and $u \in \mathscr{E}'$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{E}$.*

Proof. The proof is very similar to the proof of Lemma 13.70, so we will leave it as an exercise, see Exercise 13.72. □

Exercise 13.72. Show that, if $\phi_j \to \phi_\infty$ in $\mathscr{E}$ and $u \in \mathscr{E}'$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{E}$. ♢

Lemma 13.73. *If $\phi_j \to \phi_\infty$ in $\mathscr{D}$ and $u \in \mathscr{E}'$, then $u * \phi_j \to u * \phi_\infty$ in $\mathscr{D}$.*

Proof. If $\phi_j \to \phi_\infty$ in $\mathscr{D}$, then there is a compact $K \subset \mathbb{R}^n$ such that $\mathrm{spt}(\phi_j) \subset K$ for all j and $\mathrm{D}^\alpha\phi_j \to \mathrm{D}^\alpha\phi_\infty$ uniformly on K for all $\alpha \in \mathbb{N}^n$. Then $\mathrm{spt}(u * \phi_j) \subset \mathrm{spt}(u) + K = \tilde{K}$ by Proposition 13.69, for all j. By Lemma 13.71, $u * \phi_j \to u * \phi_\infty$ in $\mathscr{E}$. In particular, $\mathrm{D}^\alpha(u * \phi_j) \to \mathrm{D}^\alpha(u * \phi_\infty)$ uniformly on $\tilde{K}$ for all $\alpha \in \mathbb{N}^n$. This shows that $u * \phi_j \to u * \phi_\infty$ in $\mathscr{D}$. □

Theorem 13.74 ([9, Thm.4.2.1])**.** *Let L be a linear map from $\mathscr{D} = C_c^\infty(\mathbb{R}^n)$ to $C^0(\mathbb{R}^n)$. The following statements are equivalent:*

1. *L is (sequentially) continuous, i.e., if $\phi_j \to \phi_\infty$ in $\mathscr{D}$ then $L\phi_j \to L\phi_\infty$ in $C^0(\mathbb{R}^n)$, and commutes with translations, i.e., $L[\tau_x\phi] = \tau_x L\phi$ for all $x \in \mathbb{R}^n$;*
2. *there is $u \in \mathscr{D}'$ such that $L\phi = u * \phi$ for all $\phi \in \mathscr{D}$.*

Moreover, if the above conditions are met, then the distribution u is unique, and L is in fact a continuous linear operator $\mathscr{D} \to \mathscr{E}$.

Proof. (1) ⇒ (2): Define $u : \mathscr{D} \to \mathbb{R}$ by $u[\phi] = L[\check{\phi}](0)$. Since u is the composition of the continuous linear functions $\phi \mapsto \check{\phi}$, $\psi \mapsto L[\psi]$, $f \mapsto f(0)$, then u is linear and continuous, that is, $u \in \mathscr{D}'$. Moreover,

$$\begin{aligned}u * \phi(x) &= u[\tau_x\check{\phi}] \\ &\overset{(111)}{=} u[(\tau_{-x}\phi)^\vee] \\ &= L[((\tau_{-x}\phi)^\vee)^\vee](0) \\ &= L[\tau_{-x}\phi](0) \\ &= \tau_{-x}L[\phi](0) \\ &= L[\phi](x).\end{aligned}$$

For the uniqueness, notice that, if $u_1, u_2 \in \mathscr{D}'$ are such that $L\phi = u_1 * \phi = u_2 * \phi$ for all $\phi \in \mathscr{D}$, then, for all $\phi \in \mathscr{D}$, $u_1[\phi] = u_1 * \phi(0) = u_2 * \phi(0) = u_2[\phi]$, that is, $u_1 = u_2$.

(2) ⇒ (1): By Proposition 13.50, we know that $\phi \mapsto u * \phi$ is a linear map $\mathscr{D} \to \mathscr{E}$. Next, by Lemma 13.70, we know that this map is continuous. Finally, the fact that this map commutes with translations has been proven in (116) (which was an exercise). □

STEP 3: If $u_1, u_2 \in \mathscr{D}'$ are distributions, one of which has compact support, then, for every $\phi \in \mathscr{D}$,

$$L[\phi] = u_1 * (u_2 * \phi) \tag{121}$$

is a well defined element of $\mathscr{E}$ and the map $\phi \mapsto L\phi$ is continuous. Indeed, we have two cases:

1. If $u_1 \in \mathscr{E}'$, then $u_2 \in \mathscr{D}'$; so, $u_2 * \phi$ is defined by (113), $u_2 * \phi \in \mathscr{E}$ by Proposition 13.50, and thus (121) is defined by (120). Moreover, if $\phi_j \to \phi_\infty$ in $\mathscr{D}(\Omega)$, then $u_2 * \phi_j \to u_2 * \phi_\infty$ in $\mathscr{E}$ by Lemma 13.70, and thus $u_1 * (u_2 * \phi_j) \to u_1 * (u_2 * \phi_\infty)$ in $\mathscr{E}$ by Lemma 13.71.

(2) If $u_2 \in \mathscr{E}'$, then $u_1 \in \mathscr{D}'$; so, $u_2 * \phi$ is defined by (120) and $u_2 * \phi \in \mathscr{D}$ by Proposition 13.69, and thus (121) is defined by (113). Moreover, if $\phi_j \to \phi_\infty$ in $\mathscr{D}(\Omega)$, then $u_2 * \phi_j \to u_2 * \phi_\infty$ in $\mathscr{D}$ by Lemma 13.73, and thus $u_1 * (u_2 * \phi_j) \to u_1 * (u_2 * \phi_\infty)$ in $\mathscr{E}$ by Lemma 13.70.

We conclude that, by Theorem 13.74, there is a unique $u \in \mathscr{D}'$ such that $L\phi = u * \phi$ for all $\phi \in \mathscr{D}$. So, we have the definition: If $u_1, u_2 \in \mathscr{D}'$ one of which has compact support, then

$$u = u_1 * u_2 \in \mathscr{D}' \quad \Leftrightarrow \quad \forall \phi \in \mathscr{D},\ u * \phi = u_1 * (u_2 * \phi).$$

§13.30. **Properties of convolutions.** We summarize the properties of convolutions.

Proposition 13.75. *(1) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, then $u_1 * u_2 \in \mathscr{D}'$ is well defined.*
(2) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, then $u_1 * u_2 = u_2 * u_1$.
(3) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, then $\mathrm{spt}(u_1 * u_2) \subset \mathrm{spt}(u_1) + \mathrm{spt}(u_2)$.
(4) If If $u_1, u_2, u_3 \in \mathscr{D}'$, two of which have compact support, then $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$.
(5) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, and $\alpha \in \mathbb{N}^n$, then $\mathrm{D}^\alpha(u * v) = (\mathrm{D}^\alpha u) * v = u * (\mathrm{D}^\alpha v)$.
(6) If $u_1, u_2 \in \mathscr{D}'$, one of which has compact support, and one of which is smooth, then $u_1 * u_2 \in \mathscr{E}$.

Proof. The proof is left as an exercise. I only write the proof of $\mathrm{spt}(u_1 * u_2) \subset \mathrm{spt}(u_1) + \mathrm{spt}(u_2)$.

Since convolution is commutative, we can assume $u_2 \in \mathscr{E}'$. Take $\phi \in \mathscr{D}$ such that $\mathrm{spt}(\phi) \cap (\mathrm{spt}(u_1) + \mathrm{spt}(u_2)) = \emptyset$. Then $(u_1 * u_2)[\phi] \overset{(114)}{=} (u_1 * u_2) * \check{\phi}(0) = u_1 * (u_2 * \check{\phi})(0)$ where $\mathrm{spt}(u_2 * \check{\phi}) \overset{(115)}{\subset} \mathrm{spt}(u_2) + \mathrm{spt}(\check{\phi}) = \mathrm{spt}(u_2) - \mathrm{spt}(\phi)$. It follows that $\mathrm{spt}(u_1 * (u_2 * \check{\phi})) \overset{(115)}{\subset} \mathrm{spt}(u_1) + \mathrm{spt}(u_2 * \check{\phi}) \subset \mathrm{spt}(u_1) + \mathrm{spt}(u_2) - \mathrm{spt}(\phi)$. Since $\mathrm{spt}(\phi) \cap (\mathrm{spt}(u_1) + \mathrm{spt}(u_2)) = \emptyset$, then $0 \notin \mathrm{spt}(u_1 * (u_2 * \check{\phi}))$, that is $u_1 * (u_2 * \check{\phi})(0) = 0$. □

Exercise 13.76. Prove Proposition 13.75. Some parts have already been proven, others instead have been proven only partially. Put all the pieces together. ♢

Exercise 13.77. Show that $\delta_0 * u = u$ for every $u \in \mathscr{D}'$. What is $\delta_v * u$? ♢

§13.31. **Singular support.** The singular support of a distribution $u \in \mathscr{D}'(\Omega)$, denote by $\mathrm{singSpt}(u)$, is the set defined by:

$$\Omega \setminus \mathrm{singSpt}(u) = \bigcup \{U \subset \Omega \text{ open, such that } u|_V \in C^\infty(V)\}.$$

Lemma 13.78. *Let $u \in \mathscr{D}'(\Omega)$. The restriction of u to $\Omega \setminus \mathrm{singSpt}(u)$ is a smooth function.*

Exercise 13.79. Show that $\mathrm{singSpt}(u) \subset \mathrm{spt}(u)$. ♢

Saying that a distribution u is smooth is equivalent to say that $\mathrm{singSpt}(u) = \emptyset$. The smoothness statements for convolutions in Proposition 13.50 and Proposition 13.69, generalize in the following statement:

Proposition 13.80. *If $u, v \in \mathscr{D}'$, one of which has compact support, then*

$$\mathrm{singSpt}(u * v) \subset \mathrm{singSpt}(u) + \mathrm{singSpt}(v). \tag{122}$$

Proof. Let $A, B \subset \mathbb{R}^n$ open such that $\mathrm{singSpt}(u) \subset A$ and $\mathrm{singSpt}(v) \subset B$. Then there are smooth functions $a, b \in C^\infty(\mathbb{R}^n)$ such that

$$\mathrm{singSpt}(u) \subset \mathrm{int}\{a = 1\} \subset \mathrm{spt}(a) \subset A,$$
$$\mathrm{singSpt}(v) \subset \mathrm{int}\{b = 1\} \subset \mathrm{spt}(b) \subset B.$$

Then

$$u * v = ((au) * (bv)) + ((1 - au) * (bv)) + ((au) * (1 - bv)) + ((1 - au) * (1 - bv)).$$

Notice that all these convolutions are well defined. Moreover, in all but the first one, one of the factors of the convolutions is a smooth function. Hence,

$$\mathrm{singSpt}(u * v) \subset \mathrm{singSpt}((au) * (bv)) \subset \mathrm{spt}((au) * (bv)) \subset \mathrm{spt}(au) + \mathrm{spt}(bv) \subset A + B.$$

Since A and B are arbitrary, we obtain (122). □

Exercise 13.81. Show that, if $E \in \mathscr{D}'$ is such that $\mathrm{singSpt}(E) \subset \{0\}$, then $\mathrm{singSpt}(E * u) \subset \mathrm{singSpt}(u)$ for all u that can be convoluted with E. ♢

§13.32. **Linear differential operators with constant coefficients.** A *linear differential operator with constant coefficients of order* $m \in \mathbb{N}$ is a differential operator of the form

$$P = \sum_{|\alpha| \le m} c_\alpha \mathrm{D}^\alpha,$$

with some $\{c_\alpha\}_\alpha \subset \mathbb{C}$. Our four PDE are of this type:

$$\partial_t - b \cdot \nabla, \qquad -\triangle, \qquad \partial_t - \triangle, \qquad \partial_t^2 - \triangle.$$

Such a differential operator defines a linear operator $P : \mathscr{D}' \to \mathscr{D}'$. Suddenly, we can try to solve in $u \in \mathscr{D}'$ the equation

$$Pu = f$$

for some $f \in \mathscr{D}'$.

§13.33. **Fundamental solution.** A *fundamental solution* of a linear differential operator with constant coefficients P is a distribution $E \in \mathscr{D}'$ such that

$$PE = \delta_0,$$

where δ_0 is the Dirac delta centered at 0.

Exercise 13.82. Show that the function $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$,

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log(|x|) & \text{if } n = 2, \\ \frac{1}{n(n-2)\omega_n} \frac{1}{|x|^{n-2}} & \text{if } n \ge 3, \end{cases}$$

where ω_n is the volume of the unit ball in $\mathbb{R}^n$, is a fundamental solution of $P = -\triangle$. ♢

Exercise 13.83. Show that the function $\Phi : \mathbb{R}^n \times \mathbb{R} \setminus \{(0,0)\} \to [0, +\infty)$ defined by

$$\Phi(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x|^2}{4t}\right) & \text{for } x \in \mathbb{R}^n \text{ and } t > 0, \\ 0 & \text{otherwise, i.e., } (x,t) \in (\mathbb{R}^n \times (-\infty, 0]) \setminus \{(0,0)\}, \end{cases}$$

is a fundamental solution of $P = \partial_t - \triangle$. ♢

Exercise 13.84. Find a fundamental solution for the wave operator $P = \square = \partial_t^2 - \triangle$ in dimension 1, 2 and 3. ♢

§13.34. **Use of fundamental solutions.** A fundamental solution is useful for the following very simple reason. If P is a differential operator and $E \in \mathscr{D}'$ is such that $PE = \delta_0$, then, for every $f \in \mathscr{E}'$,

$$f = f * \delta_0 = f * PE = P(f * E).$$

So, $f * E$ is a solution to $Pu = f$.

§13.35. **Hypoellipticity.** A linear operator P is *hypoelliptic* if, for every $\Omega \subset \mathbb{R}^n$ open and $u \in \mathscr{D}'(\Omega)$, $Pu \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$.

Theorem 13.85. *Let P be a linear differential operator with constant coefficients. Then the following are equivalent:*

(i) Some fundamental solution E of P has $\mathrm{singSpt}(E) \subset \{0\}$.
(ii) Every fundamental solution E of P has $\mathrm{singSpt}(E) \subset \{0\}$.
(iii) P is hypoelliptic.

Proof. The implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ are clear. We need to show $(i) \Rightarrow (iii)$. Let $\Omega \subset \mathbb{R}^n$ open and $u \in \mathscr{D}'(\Omega)$ with $Pu \in C^\infty(\Omega)$. Fix $x \in \Omega$ and let $\psi \in C_c^\infty(\Omega)$ such that $x \in \operatorname{int}\{\psi = 1\}$. Then $\psi u \in \mathscr{E}'$. Moreover, by the Generalized Leibniz Rule (107), $P(\psi u) = \psi P u + R$ where $\operatorname{spt}(R) \subset \operatorname{spt}(\mathrm{D}\psi)$. So

$$\psi u = \delta_0 * (\psi u) = (PE) * (\psi u) = E * (P(\psi u)) = E * (\psi P u) + E * R.$$

and

$$\begin{aligned} \operatorname{singSpt}(\psi u) &\subset \operatorname{singSpt}(E * (\psi P u)) \cup \operatorname{singSpt}(E * R) \\ &\subset \emptyset \cup (\operatorname{singSpt}(E) + \operatorname{singSpt}(R)) \\ &\subset \operatorname{spt}(R) \subset \operatorname{spt}(\mathrm{D}\psi). \end{aligned}$$

Since $x \notin \operatorname{spt}(\mathrm{D}\psi)$, we obtain that $x \notin \operatorname{singSpt}(\psi u)$. In other words, ψu is smooth in a neighborhood of x. Since ψ is 1 in a neighborhood of x, we get that u is smooth in a neighborhood of x. We conclude that u is smooth in x. □

§13.36. Extra topics that are not covered.

- kernel theorem,
- convolution operators,
- existence of fundamental solution

14. Schwartz distributions and Fourier transform

§14.1. Schwartz test functions. The *Schwartz space* $\mathscr{S}$ of *Schwartz functions* is the space of functions $f : \mathbb{R}^n \to \mathbb{C}$ that are smooth and such that, for every $\alpha \in \mathbb{N}^n$ and every $N \in \mathbb{N}$ there exists $C_{\alpha,N}$ with

$$|\mathrm{D}^\alpha f(x)| \le C_{\alpha,N}(1+|x|)^{-N}, \qquad \forall x \in \mathbb{R}^n. \tag{123}$$

In other words, a smooth function f belongs to $\mathscr{S}$ if f and all its derivatives decrease at infinity faster than any (inverse of) polynomial. An example is $f(x) = \exp(-|x|^2)$.

The condition (123) can be expressed in different ways. For instance, one can require the upper bound $|\mathrm{D}^\alpha f(x)| \le C_{\alpha,N}(1+|x|^2)^{-N/2}$, or $|\mathrm{D}^\alpha f(x)| \le C_{\alpha,N}(1+|x|^2)^{-N}$.

Another way to express the condition (123) is as follows: a function $f \in C^\infty(\mathbb{R}^n)$ belongs to $\mathscr{S}$ if and only if, for every $\alpha, \beta \in \mathbb{N}^n$, the seminorms

$$p_{\alpha,\beta}(f) = \sup_{x\in\mathbb{R}^n} |x^\alpha \mathrm{D}^\beta f(x)|$$

are finite. An equivalent family of seminorms is

$$p_{\alpha,N}(f) = \sup_{x\in\mathbb{R}^n} (1+|x|)^N |\mathrm{D}^\alpha f(x)|.$$

We endow $\mathscr{S}$ with the family of seminorms $\{p_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^n}$, so that $\mathscr{S}$ becomes a Fréchet space. This means in particular that convergence in $\mathscr{S}$ is defined as follows:

$$f_j \to f_\infty \text{ in } \mathscr{S} \quad \Leftrightarrow \quad \forall \alpha,\beta \in \mathbb{N}^n \ \lim_{j\to\infty} p_{\alpha,\beta}(f_j - f_\infty) = 0.$$

Notice that, set-wise,

$$\mathscr{D} \subset \mathscr{S} \subset \mathscr{E}.$$

As usual, one can check that these embeddings are continuous.

Exercise 14.1. Show that $\mathscr{S} \subset L^1$.

Hint: Use (123) for N large enough. ◊

Exercise 14.2. Show that $f_j \to f_\infty$ in $\mathscr{S}$ if and only if $x^\alpha \mathrm{D}^\beta f_j \to x^\alpha \mathrm{D}^\beta f_\infty$ in L^∞, for every $\alpha, \beta \in \mathbb{N}^n$. ◊

Exercise 14.3. Show that, if $f_j \to f_\infty$ in $\mathscr{S}$, then, for every $\alpha \in \mathbb{N}^n$, $\mathrm{D}^\alpha f_j \to \mathrm{D}^\alpha f_\infty$ in $L^1(\mathbb{R}^n)$.

Hint: Take first $\alpha = 0$. Up to substituting f_j with $f_j - f_\infty$, we can also assume $f_\infty = 0$. So we need to show that, if $f_j \to 0$ in $\mathscr{S}$, then $f_j \to 0$ in $L^1(\mathbb{R}^n)$. The convergence $f_j \to 0$ in $\mathscr{S}$ implies that, for every $N > 0$ and every $\epsilon > 0$, there is $J \in \mathbb{N}$ such that $|f_j(x)|(1+|x|)^N < \epsilon$ for all $x \in \mathbb{R}^n$ and all $j > J$. ◊

Exercise 14.4. Define $g_\epsilon : \mathbb{R}^n \to [0,+\infty)$ by

$$g_\epsilon(z) = \frac{1}{\epsilon^n} e^{-\pi|z|^2/\epsilon^2} = \frac{1}{\epsilon^n}\exp(-\pi|z|^2/\epsilon^2).$$

Show that, if $f \in \mathscr{S}$, then $f * g_\epsilon \to f$ in $\mathscr{S}$ as $\epsilon \to 0$.

Hint: Go back to the proof of the first statement in Proposition 13.57. The now g_ϵ does not have compact support, but $\int_{\mathbb{R}^n\setminus B(0,1)} g_\epsilon(x)\,\mathrm{d}x$ is arbitrarily small as $\epsilon \to 0$. ◊

§14.2. Fourier transform. For $u \in L^1(\mathbb{R}^n)$ and $\xi \in \mathbb{R}^n$, define the *Fourier transform*

$$\hat{u}(\xi) = \mathscr{F}(u)(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} u(x)\,\mathrm{d}x. \tag{124}$$

Since $|e^{-2\pi i x\cdot\xi}u(x)| \le |u(x)|$, we readily have that $\hat{u}(x)$ is well defined for every $x \in \mathbb{R}^n$ and that

$$\|\hat{u}\|_{L^\infty} \le \|u\|_{L^1}. \tag{125}$$

In particular, we have that, if $u_j \to u_\infty$ in L^1, then $\hat{u}_j \to \hat{u}_\infty$ uniformly on $\mathbb{R}^n$.

§14.3. Literature on the Fourier transform. There are a lot of resources on the Fourier transforms. We follow in particular these three:

(1) the short account in Section 0.D, page 14, in G. B. Folland. *Introduction to partial differential equations.* Second. Princeton University Press, Princeton, NJ, 1995, pp. xii+324;
(2) Chapter VII in L. Hörmander. *The analysis of linear partial differential operators. I.* vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1983, pp. ix+391;
(3) Chapter 7 in W. Rudin. *Functional analysis.* Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424.

Be aware that the Fourier transform has slightly different definitions in the literature. Beside our formula (124), there are also

$$\int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,\mathrm{d}x,$$
$$\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-2\pi i x\cdot\xi} u(x)\,\mathrm{d}x.$$

These differences will lead to differences in the formulas, usually in the multiplicative constants. It is good to do the proofs with one formula, but also to try the others.

§14.4. First properties of the Fourier transform.

Exercise 14.5. Prove the following properties:

(1) If $u \in L^1(\mathbb{R}^n)$, $a \in \mathbb{R}^n$, and $u_a(x) = u(x+a)$ then $\mathscr{F}(u_a)(\xi) = e^{2\pi i a\cdot\xi}\mathscr{F}(u)(\xi)$.
(2) If $u \in L^1(\mathbb{R}^n)$, $T : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible, then

$$\mathscr{F}(u \circ T)(\xi) = |\det T|^{-1}\mathscr{F}(u)((T^{-1})^*\xi). \tag{126}$$

(3) If T is a rotation of $\mathbb{R}^n$, then $\mathscr{F}(u \circ T) = \mathscr{F}(u) \circ T$.

Hint: See [7, Proposition (0.21)] ♢

Exercise 14.6. Compute $\mathscr{F}(\bar{u})$ in terms of $\mathscr{F}(u)$. (Here $\bar{\cdot}$ denotes the complex conjugate, that is, for $x, y \in \mathbb{R}$, $\overline{x+iy} = x - iy$.) ♢

Exercise 14.7. Compute $\mathscr{F}(u(-x))$ in terms of $\mathscr{F}(u)$. ♢

Exercise 14.8. Fix $f \in \mathscr{S}$ and define $g(x) = \overline{f(-x)}$. Show that $\hat{g}(\xi) = \overline{\hat{f}(\xi)}$. ♢

Exercise 14.9. Compute the Fourier transform of the function

$$u(x) = Ae^{-a|x|^2},$$

for every $A \in \mathbb{C}$ and $a > 0$.

Solution: Let's start with $n = 1$, $A = 1$ and $a = \pi$, that is, $u(x) = e^{-\pi x^2}$. Then, for $\xi \in \mathbb{R}$,

$$\hat{u}(\xi) = \int_{\mathbb{R}} e^{-2\pi i\xi x} e^{-\pi x^2}\,\mathrm{d}x = \int_{\mathbb{R}} e^{-2\pi i\xi x - \pi x^2}\,\mathrm{d}x.$$

Since $(x+i\xi)^2 = x^2 - \xi^2 + 2ix\xi$, then $-2\pi i\xi x - \pi x^2 = -\pi(x^2 + 2ix\xi) = -\pi((x+i\xi)^2 + \xi^2)$ and thus

$$\hat{u}(\xi) = \int_{\mathbb{R}} e^{-\pi((x+i\xi)^2+\xi^2)}\,\mathrm{d}x = e^{-\pi\xi^2}\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\,\mathrm{d}x.$$

Now we see that we are integrating the holomorphic function $f(z) = e^{-\pi z^2}$ along the curve $\gamma(t) = t + i\xi$. Applying Cauchy's theorem, we can integrate along the other curve $\eta(t) = t$, and thus

$$\hat{u}(\xi) = e^{-\pi\xi^2}\int_{\mathbb{R}} e^{-\pi x^2}\,\mathrm{d}x = e^{-\pi\xi^2},$$

because $\int_{\mathbb{R}} e^{-\pi x^2}\,\mathrm{d}x = 1$. So, $\hat{u} = u$, or $\mathscr{F}(u) = u$.

We extend this result to $\mathbb{R}^n$ for $n \geq 1$:

$$\begin{aligned}\mathscr{F}(e^{-\pi|x|^2})(\xi) &= \int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x}e^{-\pi|x|^2}\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} e^{-2\pi i\sum_j \xi_j x_j - \pi\sum_j x_j^2}\,\mathrm{d}x \\ &= \prod_{j=1}^n \int_{\mathbb{R}} e^{-2\pi i\xi_j x_j - \pi x_j^2}\,\mathrm{d}x_j \\ &= \prod_{j=1}^n e^{-\pi\xi_j^2} = e^{-\pi|\xi|^2}.\end{aligned}$$

Again, we have,

$$\mathscr{F}(e^{-\pi|x|^2}) = e^{-\pi|\xi|^2},$$

that is, the function $u(x) = e^{-\pi|x|^2}$ is a fixed point of $\mathscr{F} : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$.

We now go back to the general $u : \mathbb{R}^n \to \mathbb{R}$, $u(x) = Ae^{-a|x|^2}$. By linearity of $\mathscr{F}$, we have $\mathscr{F}(Ae^{-a|x|^2}) = A\mathscr{F}(e^{-a|x|^2})$. We apply (126) with $Tx = \sqrt{a/\pi}x$, so that

$$\begin{aligned}\mathscr{F}(e^{-a|x|^2})(\xi) &= \mathscr{F}(e^{-\pi|x|^2} \circ T)(\xi) \\ &= (a/\pi)^{-n/2}\mathscr{F}(e^{-\pi|x|^2})(\xi/\sqrt{a/\pi}) \\ &= \left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{\pi^2}{a}|x|^2}.\end{aligned}$$

We now wrap up the solution to get

$$\mathscr{F}(Ae^{-a|x|^2}) = A\left(\frac{\pi}{a}\right)^{n/2} e^{-\frac{\pi^2}{a}|x|^2}. \tag{127}$$

♢

Recall from Exercise 13.42, that the convolution of two functions $f, g \in L^1(\mathbb{R}^n)$ is a well defined function $f * g \in L^1(\mathbb{R}^n)$.

Exercise 14.10. Show that, for every $f, g \in L^1(\mathbb{R}^n)$,

$$\mathscr{F}(f * g) = \mathscr{F}(f) \cdot \mathscr{F}(g). \tag{128}$$

Or, otherwise stated, $(f * g)^\wedge = \hat{f}\hat{g}$. ♢

Exercise 14.11. Show that, for every $f, g \in L^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\hat{g}(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} \hat{f}(\xi)g(\xi)\,\mathrm{d}\xi.$$

Hint: Notice that, by (125), $f\hat{g} \in L^1(\mathbb{R}^n)$ and $\hat{f}, g \in L^1(\mathbb{R}^n)$. So, unpack the definition of $\hat{g}$ and use Fubini. ♢

§14.5. The Fourier transform preserves the Schwartz class.

Proposition 14.12. *If $f \in \mathscr{S}$, then $\hat{f} \in \mathscr{S}$ and, for every $\alpha \in \mathbb{N}^n$,*

$$\mathscr{F}(\mathrm{D}^\alpha f) = (2\pi i\xi)^\alpha \mathscr{F}(f), \tag{129}$$

$$\mathscr{F}((-2\pi i x)^\alpha f) = \mathrm{D}^\alpha \mathscr{F}(f). \tag{130}$$

Proof. For $f \in \mathscr{S}$ and $j \in \{1, \dots, n\}$, we have $\partial_j f \in \mathscr{S}$ and thus

$$\begin{aligned}\mathscr{F}(\partial_j f)(\xi) &= \int_{\mathbb{R}^n} \exp(-2\pi i\xi\cdot x)\frac{\partial}{\partial x_j}f(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial x_j}[\exp(-2\pi i\xi\cdot x)f(x)]\,\mathrm{d}x - \int_{\mathbb{R}^n}(-2\pi i\xi_j)\exp(-2\pi i\xi\cdot x)f(x))\,\mathrm{d}x \\ &= (2\pi i\xi_j)\hat{f}(\xi).\end{aligned}$$

Iterating, we obtain (129).

Since $\mathscr{F}(\mathrm{D}^\alpha f) \in L^\infty(\mathbb{R}^n)$ for every $\alpha \in \mathbb{N}^n$, we conclude that $\hat{f}$, which a priori only belongs to $L^\infty(\mathbb{R}^n)$, in fact satisfies

$$p_{0,N}(\hat{f}) = \sup_{\xi\in\mathbb{R}^n} (1+|\xi|)^N |\hat{f}(\xi)| < \infty, \tag{131}$$

for every $N \in \mathbb{N}$.

We don't know yet that $\hat{f}$ is differentiable. For this, we use **(3.3.3)** in Theorem 3.3 to compute, for $f \in \mathscr{S}$ and $j \in \{1,\dots,n\}$,

$$\begin{aligned}\partial_j \mathscr{F}(f)(\xi) &= \frac{\partial}{\partial \xi_j}\int_{\mathbb{R}^n} e^{-2\pi i\xi\cdot x} f(x)\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} \frac{\partial}{\partial \xi_j}[e^{-2\pi i\xi\cdot x} f(x)]\,\mathrm{d}x \\ &= \int_{\mathbb{R}^n} (-2\pi i x_j) e^{-2\pi i\xi\cdot x} f(x)\,\mathrm{d}x \\ &= \mathscr{F}((-2\pi i x_j) f)(\xi).\end{aligned}$$

Iterating, we obtain (130). We conclude that $\hat{f}$ is smooth and that, for every $\alpha \in \mathbb{N}^n$, since $\mathrm{D}^\alpha \hat{f}$ is the Fourier transform of a function in $\mathscr{S}$, then, by (131),

$$p_{\alpha,N}(\hat{f}) = p_{0,N}(\mathrm{D}^\alpha \hat{f}) < \infty.$$

We conclude that $\hat{f} \in \mathscr{S}$. □

Corollary 14.13 (Riemann–Lebesgue Lemma)**.** *If $f \in L^1(\mathbb{R}^n)$, then $\hat{f} \in C^0_0(\mathbb{R}^n)$, that is, $\hat{f}$ is continuous and tends to zero at infinity.*

Proof. We know that $\mathscr{S}$ is dense in $L^1(\mathbb{R}^n)$, see Exercise 14.14. Let $f \in L^1(\mathbb{R}^n)$ and $f_j \in \mathscr{S}$ a sequence with $f_j \to f$ in $L^1(\mathbb{R}^n)$. By (125), $\hat{f}_j \to \hat{f}$ in $L^\infty(\mathbb{R}^n)$, that is $\hat{f}_j \to \hat{f}$ uniformly. It follows that $\hat{f}$ is continuous.

We need to show that $\hat{f}$ tends to zero at infinity. For every $\epsilon > 0$ there is $j \in \mathbb{N}$ with $\|f_j - f\|_{L^\infty} < \epsilon$. Then, there is $R > 0$ such that $|f_j(\xi)| < \epsilon$ for every ξ with $|\xi| > R$. Therefore, if $|\xi| > R$, then $|f(\xi)| \le |f(\xi) - f_j(\xi)| + |f_j(\xi)| < 2\epsilon$. This shows that $\lim_{\xi\to\infty} \hat{f}(\xi) = 0$. □

Exercise 14.14. Show that $\mathscr{S}$ is dense in $L^1(\mathbb{R}^n)$.

Hint: Given $f \in L^1(\mathbb{R}^n)$, consider $f_j(x) = \psi(x/j) f * \rho_{1/j}(x)$, where $\{\rho_\epsilon\}_{\epsilon>0}$ is a family of mollifiers, and $\psi \in C^\infty_c(\mathbb{R}^n)$ is a function valued in $[0,1]$ with $B(0,1) \subset \{\psi = 1\}$. You then need to show that $f_j \in \mathscr{S}$ and that $f_j \to f$ in $L^1(\mathbb{R}^n)$. Use the fact that, for every $\epsilon > 0$ there exists $R > 0$ such that $\int_{\mathbb{R}^n\setminus B(0,R)} |f(x)|\,\mathrm{d}x < \epsilon$ (this is a direct consequence of integrability). ♢

§14.6. Fourier inversion theorem.

Theorem 14.15. *The Fourier transform $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is a linear automorphism of $\mathscr{S}$. In particular, if $f \in \mathscr{S}$, then*

$$\mathscr{F}^{-1}(f)(x) = \int_{\mathbb{R}^n} e^{2\pi i x\cdot\xi} f(\xi)\,\mathrm{d}\xi. \tag{132}$$

Proof. We know that $\mathscr{F}$ is linear. We need to show that it is onto and into, and that both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are continuous.

First we show that $\mathscr{F}$ is continuous. If $\alpha, \beta \in \mathbb{N}^n$, then

$$\begin{aligned}\xi^\alpha \mathrm{D}^\beta \hat{f}(\xi) &\overset{(130)}{=} \xi^\alpha \mathscr{F}((-2\pi i x)^\beta f) \\ &\overset{(129)}{=} \frac{1}{(2\pi i)^\alpha}\mathscr{F}(\mathrm{D}^\alpha(-2\pi i x)^\beta f) \\ &= \frac{(-2\pi i)^\beta}{(2\pi i)^\alpha}\mathscr{F}(\mathrm{D}^\alpha(x^\beta f)).\end{aligned}$$

If $f_j \to 0$ in $\mathscr{S}$, then $\mathrm{D}^\alpha(x^\beta f_j) \to 0$ in $\mathscr{S}$, for every $\alpha, \beta \in \mathbb{N}^n$. It follows that $\mathrm{D}^\alpha(x^\beta f_j) \to 0$ in $L^1(\mathbb{R}^n)$, and thus $\xi^\alpha \mathrm{D}^\beta \hat{f}_j \to 0$ in $L^\infty(\mathbb{R}^n)$, for every $\alpha, \beta \in \mathbb{N}^n$. This means that $\hat{f}_j \to 0$ in $\mathscr{S}$ (see Exercise 14.2). So, $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is continuous.

Denote by $\mathscr{G}(f)(x)$ the quantity on the right-hand side of (132). Notice that $\mathscr{G}(f)(x) = \mathscr{F}(f)(-x)$: so, we know that $\mathscr{G}$ is a continuous linear operator $\mathscr{S} \to \mathscr{S}$. We will show that $\mathscr{G}(\mathscr{F}(f)) = f$, for all $f \in \mathscr{S}$.

For $x \in \mathbb{R}^n$ and $\epsilon > 0$, define

$$\phi_{x,\epsilon}(\xi) = e^{2\pi i x\cdot\xi - \pi\epsilon^2|\xi|^2} = \exp(2\pi i x \cdot \xi - \pi\epsilon^2|\xi|^2).$$

If we set $g_\epsilon(z) = e^{-\pi|z|^2/\epsilon^2} = \exp(-\pi|z|^2/\epsilon^2)$, then one can compute

$$\hat{\phi}_{x,\epsilon}(y) = g_\epsilon(x-y).$$

Then we have

$$\begin{aligned}
\mathscr{G}(e^{-\pi\epsilon^2|\xi|^2}\hat{f})(x) &= \int_{\mathbb{R}^n} e^{-\pi\epsilon^2|\xi|^2} e^{2\pi i x\cdot\xi}\hat{f}(\xi)\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \phi_{x,\epsilon}(\xi)\hat{f}(\xi)\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \hat{\phi}_{x,\epsilon}(y) f(y)\,\mathrm{d}y \\
&= \int_{\mathbb{R}^n} g_\epsilon(x-y) f(y)\,\mathrm{d}y \\
&= g_\epsilon * f(x).
\end{aligned}$$

On the one hand, $g_\epsilon * f \to f$ in $\mathscr{S}$ as $\epsilon \to 0$ by Exercise 14.4, hence point-wise. On the other hand, $e^{-\pi\epsilon^2|\xi|^2}\hat{f} \to \hat{f}$ in $L^1(\mathbb{R}^n)$ as $\epsilon \to 0$. Therefore, using also the continuity of $\mathscr{G}$, we conclude that $\mathscr{G}(\hat{f}) = f$.

We complete the proof with a bit of algebra. Define $\mathcal{I} : \mathscr{S} \to \mathscr{S}$, $\mathscr{I}(f)(x) = f(-x)$. We have just proven

$$\mathscr{F} \circ \mathcal{I} \circ \mathscr{F} = \mathrm{Id}_{\mathscr{S}}.$$

This shows that $\mathscr{F}$ is also surjective, with inverse $\mathscr{G} = \mathcal{I} \circ \mathscr{F}$. □

Remark 14.16. Notice that, if you write down the formula for $\mathscr{G}(\mathscr{F}(f))$, you cannot conclude using Fubini! In fact, we have somehow proved that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{2\pi i(x-y)\cdot\xi} f(y)\,\mathrm{d}y\,\mathrm{d}\xi = f(x).$$

This identity does not have a meaning as integrals, because the integrand function is not in $L^1(\mathbb{R}^n \times \mathbb{R}^n)$. However, it does look a lot the convolution of f with δ_0, in which case we would have proven

$$\int_{\mathbb{R}^n} e^{2\pi i z\cdot\xi}\,\mathrm{d}\xi = \delta_0(z). \tag{133}$$

In the latter identity, the integral is not an integral: we will make sense of (133) distributionally.

Remark 14.17. The inverse of the Fourier transform applied to f is also denoted as $\check{f} = \mathscr{F}^{-1}(f)$. This is a problem in my notes, because I have already used this notation for the function $\check{f}(x) = f(-x)$. This clash of notation does not have a solution yet. I am just renouncing to use the second meaning in this section, so that, in the context of Fourier transform, $\check{f}$ is only the inverse transform of f.

Exercise 14.18. Define $\mathcal{I} : \mathscr{S} \to \mathscr{S}$, $\mathcal{I}(f)(x) = f(-x)$. Show that

$$\mathscr{F}^2 = \mathcal{I} \quad \text{and} \quad \mathscr{F}^4 = \mathrm{Id}_{\mathscr{S}}.$$

Hint: Forget about the Fourier transform, but only use $\mathscr{F}\mathcal{I}\mathscr{F} = 1$. ◊

§14.7. **Plancherel Theorem.** The space $L^2(\mathbb{R}^n)$ is the space of complex valued measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that $\int_{\mathbb{R}^n} |f(x)|^2 \,\mathrm{d}x < \infty$. The vector space $L^2(\mathbb{R}^n)$ is a Hilbert space when endowed with the sesquilinear form

$$\langle f, g\rangle = \int_{\mathbb{R}^n} f(x)\bar{g(x)}\,\mathrm{d}x, \quad \forall f, g \in L^2(\mathbb{R}^n).$$

Theorem 14.19. *The Fourier transform extends uniquely to a unitary automorphism of $L^2(\mathbb{R}^n)$. More precisely, the Fourier transform has a continuous extension $\mathscr{F} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ with*

$$\|\mathscr{F}(f)\|_{L^2} = \|f\|_{L^2} \quad \forall f \in L^2(\mathbb{R}^n),$$
$$\langle \mathscr{F}(f), \mathscr{F}(g)\rangle = \langle f, g\rangle \quad \forall f, g \in L^2(\mathbb{R}^n).$$

Proof. By Exercise 14.20 and Theorem 14.15, we know that $\mathscr{F} : \mathscr{S} \to \mathscr{S}$ is continuous with respect to the topology of $L^2(\mathbb{R}^n)$. Since $\mathscr{S}$ is dense in $L^2(\mathbb{R}^n)$, then $\mathscr{F}$ admits a unique continuous extension $L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$. The inverse of $\mathscr{F}$ has the same properties, and thus the extension of $\mathscr{F}$ is a continuous invertible linear operator.

We only need to show $\|\mathscr{F}(f)\|_{L^2} = \|f\|_{L^2}$ for every $f \in \mathscr{S}$. Fix $f \in \mathscr{S}$ and define $g(x) = \overline{f(-x)}$. Notice that $\hat{g}(\xi) = \overline{\hat{f}(\xi)}$; see also Exercise 14.8. Then:

$$\begin{aligned}
\|\mathscr{F}(f)\|_{L^2} &= \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{f}(\xi)}\,\mathrm{d}\xi \\
\text{[by Exercise 14.8]} &= \int_{\mathbb{R}^n} \hat{f}(\xi)\hat{g}(\xi)\,\mathrm{d}\xi \\
&\overset{(128)}{=} \int_{\mathbb{R}^n} (f * g)^\wedge(\xi)\,\mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} e^{2\pi i 0\cdot\xi}(f * g)^\wedge(\xi)\,\mathrm{d}\xi \\
&\overset{(132)}{=} f * g(0) \\
&= \int_{\mathbb{R}^n} f(x)\overline{f(-x)}\,\mathrm{d}x \\
&= \|f\|_{L^2}.
\end{aligned}$$

□

Exercise 14.20. (Maybe this already appeared before). Show that $\mathscr{D}$ is dense in $L^p(\mathbb{R}^n)$ for all $p \in [1, \infty)$. Deduce that $\mathscr{S}$ is dense in $L^p(\mathbb{R}^n)$ for all $p \in [1, \infty)$.

But then, show also that, if $f_j \to f_\infty$ in $\mathscr{D}$ or $\mathscr{S}$, then $f_j \to f_\infty$ in $L^p(\mathbb{R}^n)$ for all $p \in [1, \infty)$. ♢

§14.8. **Schwartz distributions.** The topological dual $\mathscr{S}'$ is the space of *Schwartz distributions*, or *tempered distributions*. The convergence of sequences in $\mathscr{S}'$ is pointwise, that is, $A_n \to A_\infty$ in $\mathscr{S}'$ if and only if $A_n[\phi] \to A_\infty[\phi]$ for every $\phi \in \mathscr{S}$.

Exercise 14.21. Show that

$$\mathscr{D}' \supset \mathscr{S}' \supset \mathscr{E}'.$$

♢

Exercise 14.22. Show the following properties of tempered distributions:

(1) If $u \in \mathscr{S}'$ and $\alpha \in \mathbb{N}^n$, then $\mathrm{D}^\alpha u \in \mathscr{S}'$.
(2) If $u \in \mathscr{S}'$ and $f \in C^\infty(\mathbb{R}^n)$ is such that, for all $\alpha \in \mathbb{N}^n$, $\mathrm{D}^\alpha f$ grows at most polynomially at infinity, then $fu \in \mathscr{S}'$.
(3) If $u \in \mathscr{S}'$ and $f \in \mathscr{S}$, then $u * f \in \mathscr{S}$.

♢

Exercise 14.23. Show that, if $h : \mathbb{R}^n \to \mathbb{C}$ is a measurable function that grows at most polynomially, then $f \mapsto \int_{\mathbb{R}^n} h(x)f(x)\,\mathrm{d}x$ defines a tempered distribution. ♢

§14.9. The Fourier transform of tempered distributions. We define the Fourier transform of a tempered distribution $u \in \mathscr{S}'$ as $\hat{u} = \mathscr{F}(u)$, where

$$\hat{u}[f] = u[\hat{f}], \qquad \forall f \in \mathscr{S}.$$

Exercise 14.24. Show the following properties of the Fourier transform of tempered distributions:

(1) If $u \in \mathscr{S}'$, then $\hat{u} \in \mathscr{S}'$.
(2) If $u \in \mathscr{S}'$ is actually in $\mathscr{S}$, i.e., $u[f] = \int_{\mathbb{R}^n} u(x)f(x)\,\mathrm{d}x$ for all $f \in \mathscr{S}$, then $\hat{u}$ as distribution is equal to $\hat{u}$ as function.
(3) $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is a continuous, invertible linear operator, with inverse $\mathscr{F}^{-1}u[f] = u[\mathscr{F}^{-1}(f)]$.
(4) If $u \in \mathscr{S}'$ and $f \in \mathscr{S}$, then

$$\mathscr{F}(u * f) = \hat{u}\hat{f}. \tag{134}$$

(5) If $f \in \mathscr{S}'$, then $\hat{f} \in \mathscr{S}'$ and, for every $\alpha \in \mathbb{N}^n$,

$$\begin{aligned} \mathscr{F}(\mathrm{D}^\alpha f) &= (2\pi i\xi)^\alpha \mathscr{F}(f), \\ \mathscr{F}((-2\pi i x)^\alpha f) &= \mathrm{D}^\alpha \mathscr{F}(f). \end{aligned} \tag{135}$$

♢

Exercise 14.25. Compute $\mathscr{F}(\delta_0)$. ♢

Exercise 14.26. For $a \in \mathbb{R}^n$, compute $\mathscr{F}(\delta_a)$ and $\mathscr{F}(e^{ia\cdot x})$. Then compute $\mathscr{F}^{-1}(\delta_a)$ and $\mathscr{F}^{-1}(e^{ia\cdot x})$.

Solution. Using only definitions, we have, for all $a \in \mathbb{R}^n$ and $\phi \in \mathscr{S}$, $\mathscr{F}(\delta_a)[\phi] = \delta_a[\mathscr{F}\phi] = \mathscr{F}\phi(a) = \int_{\mathbb{R}^n} \exp(-2\pi i a\cdot x)\phi(x)\,\mathrm{d}x$. This identity exactly means that, as distributions over $\mathbb{R}^n$, $\mathscr{F}(\delta_a) = \exp(-2\pi i a\cdot x)$.

There is not much else to compute. Notice that $\mathcal{I}\delta_a[\phi] = \delta_a[\mathcal{I}\phi] = \phi(-a) = \delta_{-a}[\phi]$. So,

$$\begin{aligned} \mathscr{F}(\delta_a) &= \exp(-2\pi i a\cdot x), \\ \delta_{-a} &= \mathcal{I}\delta_a = \mathscr{F}^2(\delta_a) = \mathscr{F}_x(\exp(-2\pi i a\cdot x)), \\ \mathscr{F}_x(e^{ia\cdot x}) &= \delta_{\frac{a}{2\pi}}, \\ \mathscr{F}^{-1}\delta_a &= \exp(2\pi i a\cdot x), \\ \mathscr{F}^{-1}(e^{ia\cdot x}) &= \mathcal{I}\mathscr{F}(e^{ia\cdot x}) = \mathcal{I}\delta_{\frac{a}{2\pi}} = \delta_{-\frac{a}{2\pi}}. \end{aligned} \tag{136}$$

♢

Exercise 14.27. Compute $\mathscr{F}(1)$. ♢

Exercise 14.28. Compute $\mathscr{F}(p(x))$, where $p(x) = \sum_{|\alpha|\le N} c_\alpha x^\alpha$ is a polynomial of degree N. ♢

Remark 14.29. It is a fact that, if $\hat{u}$ has compact support, then u is analytic, see [9, Thm 7.1.14] It is a recurrent theme that regularity of u is proportional to integrability of $\hat{u}$ (and viceversa, of course). There are two sorts of "equilibrium points" of this behavior: $\mathscr{S}$ and $L^2(\mathbb{R}^n)$.

§14.10. Applications to PDE: Harmonic polynomials. If $u \in \mathscr{S}'$ is such that

$$-\triangle u = 0,$$

then

$$0 = \mathscr{F}(-\triangle u) \overset{(135)}{=} -\sum_{j=1}^n (2\pi i\xi_j)^2 \hat{u} = 4\pi^2|\xi|^2\hat{u}.$$

Therefore, $\mathrm{spt}(\hat{u}) \subset \{0\}$, that is, by Proposition 13.37, $\hat{u} = \sum_{|\alpha|\le N} c_\alpha \mathrm{D}^\alpha\delta_0$ for some constants $c_\alpha \in \mathbb{C}$. It follows that u is a polynomial:

Proposition 14.30. *Harmonic tempered distributions are harmonic polynomials.*

Exercise 14.31. The condition $|\xi|^2 v = 0$ implies $\mathrm{spt}(v) \subset \{0\}$. Can you characterize the distributions $v \in \mathscr{S}'$ that satisfy $|\xi|^2 v = 0$? ♢

§14.11. Applications to PDE: Heat equation. See also [7, §4.A, p.143] and [5, p.192].

Let's consider the heat equation

$$\begin{cases} \partial_t u - \triangle u = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u = g & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{137}$$

where we can see u as a function $[0, +\infty) \to \mathscr{S}'$, and $g \in \mathscr{S}'$.

We apply the Fourier transform in the spatial variable, so that (137) becomes

$$\begin{cases} \partial_t \hat{u} - (2\pi i)^2 |\xi|^2 \hat{u} = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ \hat{u} = \hat{g} & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

A solution to the ODE $\partial_t \hat{u} = -4\pi^2 |\xi|^2 \hat{u}$ with $\hat{u}(0) = \hat{g}$ is

$$\hat{u}(t) = \exp(-4\pi^2 |\xi|^2 t)\hat{g}.$$

Notice that, for each $t > 0$, $\xi \mapsto \exp(-4\pi^2|\xi|^2 t)$ is an element of $\mathscr{S}$, so the $\hat{u}(t) \in \mathscr{S}'$. It follows that, for $t > 0$,

$$\begin{aligned} u(t) &= \mathscr{F}_\xi^{-1}(\exp(-4\pi^2|\xi|^2 t)\hat{g}) \\ &\overset{(134)}{=} \mathscr{F}_\xi^{-1}(\exp(-4\pi^2|\xi|^2 t)) * \mathscr{F}_\xi^{-1}(\hat{g}) \\ &\overset{(127)}{=} \left(\frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}} \right) * g. \end{aligned}$$

We have obtained again a representation formula for the solution of the heat equation, as we did in Theorem 11.12.

§14.12. Applications to PDE: Wave equation. See also [7, §5.D, p.177] and [5, p.194].

Let's consider the wave equation

$$\begin{cases} \partial_t^2 u - \triangle u = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ u = g, \ \partial_t u = h & \text{on } \mathbb{R}^n \times \{0\}. \end{cases} \tag{138}$$

where we can see u as a function $[0, +\infty) \to \mathscr{S}'$, and $g, h \in \mathscr{S}'$.

We apply the Fourier transform in the spatial variable, so that (138) becomes

$$\begin{cases} \partial_t^2 \hat{u} - (2\pi i)^2 |\xi|^2 \hat{u} = 0 & \text{in } \mathbb{R}^n \times (0, +\infty), \\ \hat{u}(0) = \hat{g} \ \partial_t \hat{u}(0) = \hat{h} & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

A solution to the ODE $\partial_t^2 \hat{u} = -4\pi^2 |\xi|^2 \hat{u}$ is

$$\hat{u}(t) = \exp(2\pi i |\xi| t) A + \exp(-2\pi i |\xi| t) B.$$

The initial data $\hat{u}(0) = \hat{g}$ and $\partial_t \hat{u}(0) = \hat{h}$, give

$$\begin{aligned} \hat{u}(t) &= \frac{\exp(2\pi i|\xi|t)}{2} \left(\hat{g} + \frac{\hat{h}}{2\pi i|\xi|} \right) + \frac{\exp(-2\pi i|\xi|t)}{2} \left(\hat{g} - \frac{\hat{h}}{2\pi i|\xi|} \right) \\ &= \cos(2\pi i|\xi|t)\hat{g} + \frac{\sin(2\pi i|\xi|t)}{2\pi i|\xi|}\hat{h}. \end{aligned}$$

These formulas give another representation for the solutions of the wave equation. Since we know what they must be for $n = 1, 2, 3$, then we know that $u(t)$ must be as in **§12.6**, **§12.10** and **§12.9**, respectively.

Remark 14.32. Notice that the function $\xi \mapsto \frac{1}{|\xi|}$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ for $n \geq 2$. Since it also decays at infinity, this function is a tempered distribution. So, it has a well defined Fourier transform. You can try to compute it: I don't know what it should be, or even if there is a closed formula for it.

Exercise 14.33. Assuming $g, h \in \mathscr{S}$, write $u(t)$ from the formula

$$\hat{u}(t) = \cos(2\pi i|\xi|t)\hat{g} + \frac{\sin(2\pi i|\xi|t)}{2\pi i|\xi|}\hat{h}.$$

◊

§14.13. Applications to PDE: Bessel potentials. (See [5, §4.3, p.191])

Exercise 14.34 (Bessel Potentials)**.** Using the Fourier transform, give a representation to the solutions of

$$-\triangle u + u = f.$$

Hint: See [5, §4.3, p.191]. ◊

§14.14. Applications to PDE: Eigenfunctions of the Laplacian.

Exercise 14.35 (Eigenfunctions of the Laplacian)**.** Using the Fourier transform, give a representation to the solutions of

$$-\triangle u = \lambda u,$$

for $\lambda \in \mathbb{C}$. For which λ there exists a solution? (see also **§17.1**) ◊

Part 3. Sobolev Spaces

15. Sobolev spaces

§15.1. Definition of sobolev space. Let $\Omega \subset \mathbb{R}^n$ open. We have seen in **§13.5** that functions $u \in L^1_{\mathrm{loc}}(\Omega)$ are distributions, and as such they have all derivatives $\mathrm{D}^\alpha u$. We are interested in those functions whose distributional derivatives are in fact integrable functions.

For $m \in \mathbb{N}$ and $p \in [1, +\infty]$, define

$$W^{m,p}(\Omega) = \left\{u \in L^1_{\mathrm{loc}}(\Omega) : \mathrm{D}^\alpha u \in L^p(\Omega)\ \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq m\right\}.$$

Of these spaces, we also have a "loc" version:

$$W^{m,p}_{\mathrm{loc}}(\Omega) = \left\{u \in L^1_{\mathrm{loc}}(\Omega) : \mathrm{D}^\alpha u \in L^p_{\mathrm{loc}}(\Omega)\ \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq m\right\}.$$

§15.2. Sobolev spaces are Banach spaces. We endow $W^{m,p}(\Omega)$ with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \|u\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}.$$

In fact, we may well take in place of $\|u\|_{W^{m,p}(\Omega)}$ other norms, such as

$$\Big(\sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha u\|^q_{L^p(\Omega)}\Big)^{1/q}, \text{ for } q \in [1, +\infty), \text{ or } \max_{|\alpha| \leq m} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}.$$

See Exercise 15.1.

Exercise 15.1. For $x \in \mathbb{R}^N$ and $q \in [1, +\infty)$, define

$$\|x\|_q = \Big(\sum_{j=1}^N |x_j|^q\Big)^{1/q}, \text{ and } \|x\|_\infty = \max_{j \leq N} |x_j|.$$

Show that for every $q_1, q_2 \in [1, +\infty]$ there is L such that

$$\frac{1}{L}\|x\|_{q_2} \leq \|x\|_{q_1} \leq L\|x\|_{q_2}.$$

Deduce that all norms on $W^{m,p}(\Omega)$

$$\Big(\sum_{|\alpha| \leq m} \|\mathrm{D}^\alpha u\|^q_{L^p(\Omega)}\Big)^{1/q}, \text{ for } q \in [1, +\infty), \text{ or } \max_{|\alpha| \leq m} \|\mathrm{D}^\alpha u\|_{L^p(\Omega)}.$$

are biLipschitz equivalent to $\|u\|_{W^{m,p}(\Omega)}$. ♢

Lemma 15.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $p \in [1, \infty]$. Let $\{u_j\}_{j \in \mathbb{N}} \subset L^p(\Omega)$ be a sequence of functions, and $v, w \in L^p(\Omega)$. Fix $\ell \in \{1, \dots, n\}$. Suppose that $u_j \to v$ in $L^p(\Omega)$ and that $\partial_\ell u_j \to w$ in $L^p(\Omega)$. Then $\partial_\ell v = w$.*

Proof. First of all, we claim that $u_j \to v$ in $\mathscr{D}'(\Omega)$. Indeed, if $\phi \in \mathscr{D}(\Omega)$, then

$$\left|\int_\Omega u_j(x)\phi(x)\,\mathrm{d}x - \int_\Omega v(x)\phi(x)\,\mathrm{d}x\right| \leq \int_\Omega |u_j(x) - v(x)||\phi(x)|\,\mathrm{d}x \overset{\text{(Hölder)}}{\leq} \|u_j - v\|_{L^p(\Omega)}\|\phi\|_{L^{p'}(\Omega)} \to 0.$$

Therefore, as distributions, $u_j \to v$. It follows that $\partial_\ell u_j \to \partial_\ell v$ in $\mathscr{D}'(\Omega)$, see Exercise 13.15.

Since $\partial_\ell u_j \to w$ in $L^p(\Omega)$, we also have $\partial_\ell u_j \to w$ in $\mathscr{D}'(\Omega)$. Therefore, $\partial_\ell v = w$, for the uniqueness of the limit in $\mathscr{D}'(\Omega)$. □

Proposition 15.3. *The normed vector space $(W^{m,p}(\Omega), \|\cdot\|_{W^{m,p}(\Omega)})$ is a Banach space.*

Proof. The proof that $\|\cdot\|_{W^{m,p}(\Omega)}$ is in fact a norm is left as an exercise, see Exercise 15.4. To show that $W^{m,p}(\Omega)$ is complete, let $\{u_j\}_{j \in \mathbb{N}} \subset W^{m,p}(\Omega)$ be a Cauchy sequence. It follows that, for ever $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq m$, the sequence $\{\mathrm{D}^\alpha u_j\}_{j \in \mathbb{N}} \subset L^p(\Omega)$ is a Cauchy sequence. Since $L^p(\Omega)$ are Banach spaces, it follows that there are $u_\alpha \in L^p(\Omega)$ such that $\mathrm{D}^\alpha u_j \to u_\alpha$ in $L^p(\Omega)$. Iterating Lemma 15.2, we obtaind that $\mathrm{D}^\alpha u_0 = u_\alpha$, and thus u_0, the limit of u_j in $L^p(\Omega)$, is the limit of u_j in $W^{m,p}(\Omega)$. □

Exercise 15.4. Show that $\|\cdot\|_{W^{m,p}(\Omega)}$ is a norm on $W^{m,p}(\Omega)$. ♢

Exercise 15.5. Show that $W^{m,p}(\Omega)$ is isometric to a closed subspace of $L^p(\Omega)^N$, where $N = \#\{\alpha \in \mathbb{N}^n : |\alpha| \le m\}$. ♢

Exercise 15.6. Inspired by Lemma 15.2, we can say with no doubt that, if $u_j \to u$ in $W^{m,p}(\Omega)$, then $u_j \to u$ in $\mathscr{D}'(\Omega)$. We may wonder if the converse is also true, that is: is it true that, if $u_j \to u$ in $\mathscr{D}'(\Omega)$ and $u \in W^{m,p}(\Omega)$, then $u_j \to u$ in $W^{m,p}(\Omega)$? If this was a video, you would pause it to think about the question yourself. However, this is not a video: can you stop reading?

The answer is not hard. The point is to find a sequence of functions v_j on $\mathbb{R}$ that converge distributionally to 0, but have L^p norm equal to 1. For instance, take $v_j(x) = \sum_{m=0}^{2^j} \mathbb{1}_{[m2^{-j},(m+1)2^{-j})}(x)$. Then $|v_j| = \mathbb{1}_{[0,1)}$, and $\int_{\mathbb{R}} v_j \phi \,\mathrm{d}x \to 0$ for every $\phi \in \mathscr{D}(\mathbb{R})$ (check it!). We have thus a sequence $v_j \to 0$ in $\mathscr{D}'(\mathbb{R})$ with $\|v_j\|_{L^p} = 1$ for all j. Take $u_j(x) = \int_{-\infty}^x v_j(t)\,\mathrm{d}t$: check that $u_j(2) = 0$. So, u_j is an absolutely continuous function $\mathbb{R} \to \mathbb{R}$ with compact support. Moreover, $u_j \to 0$ distributionally (check it!) but u_j does not converge to 0 in $W^{1,1}(\mathbb{R})$. So, the answer is no. ♢

Exercise 15.7. An example of weird Sobolev functions. For $\beta \in \mathbb{R}$, define $u_\beta : \mathbb{R}^n \to \mathbb{R}$,

$$u_\beta(x) = |x|^\beta.$$

(1) Show that $u_\beta \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ whenever $\beta > -n$.
(2) Show that $u_\beta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ for all $\beta > 1 - n$.
(3) Take an enumeration $\{q_k\}_{k\in\mathbb{N}} = \mathbb{Q}^n \cap B(0,1)$ of points with rational coordinates inside the unit ball. Define

$$w_\beta(x) = \sum_{k=1}^{\infty} \frac{u_\beta(x - q_k)}{2^k}.$$

Notice that, since $u_\beta \ge 0$, then $w_\beta(x) \in [0,+\infty]$ is well defined for every $x \in \mathbb{R}^n$. Show that, if $\beta > 1 - n$, then $w_\beta \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$.

I want to remark that, if $n \ge 2$, then $1 - n < 0$ and thus we can take $\beta < 0$: in these cases, w_β has a "pole" at every point in $\mathbb{Q}^n \cap B(0,1)$. ♢

§15.3. Smooth approximation of Sobolev functions: characterization of Sobolev spaces. We are going to prove the following theorem:

Theorem 15.8. *Let $\Omega \subset \mathbb{R}^n$, $p \in [1,+\infty)$ and $m \in \mathbb{N}$. A distribution $u \in \mathscr{D}'(\Omega)$ belongs to $W^{m,p}(\Omega)$ if and only if there is a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C^\infty(\Omega) \cap W^{m,p}(\Omega)$ such that* $\lim_{j\to\infty} \|u_j - u\|_{W^{m,p}(\Omega)} = 0$.

In other words, $W^{m,p}(\Omega)$ is the closure of $C^\infty(\Omega) \cap W^{m,p}(\Omega)$ in the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.

§15.4. Smooth approximation of Sobolev functions: local approximation by convolution. Recall that we say that $w \in C^\infty(\bar\Omega)$ if there is an open set $V \subset \mathbb{R}^n$ with $\bar\Omega \subset V$, and there is a function $\tilde w \in C^\infty(V)$ such that $\tilde w|_\Omega = w$. Notice that, if Ω is bounded, then $\bar\Omega$ is compact and thus, if $w \in C^\infty(\bar\Omega)$ then $w \in W^{p,m}(\Omega)$ for all $p \in [1,+\infty]$ and $m \in \mathbb{N}$. We will show three approximation results

Theorem 15.9. *Let $\Omega \subset \mathbb{R}^n$ open. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a standard family of mollifiers. For $\epsilon > 0$, define*

$$\begin{aligned}\Omega_\epsilon &= \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) > \epsilon\} \\ &= \{x \in \Omega : \bar B(x,\epsilon) \subset \Omega\}\end{aligned}$$

For $u \in L^1_{\mathrm{loc}}(\Omega)$, define $u_\epsilon : \Omega_\epsilon \to \mathbb{C}$, $u_\epsilon = u * \rho_\epsilon$; more precisely,

$$u_\epsilon = ((u\mathbb{1}_\Omega) * \rho_\epsilon)|_{\Omega_\epsilon}. \tag{139}$$

If $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$ and $V \Subset \Omega$ is open, then $u_\epsilon \to u$ in $W^{m,p}(V)$.

Proof. Recall that $V \Subset \Omega$ means that the closure of V is compact and contained in Ω. Notice also that there is $\delta > 0$ such that $V \subset \Omega_\epsilon$ for all $\epsilon \in (0,\delta)$. So, the convergence "$u_\epsilon \to u$ in $W^{m,p}(V)$" makes sense eventually for $\epsilon \to 0$.

From Proposition 13.57, we already know that the functions u_ϵ defined in (139) converge to $u\mathbb{1}_\Omega$ in $\mathscr{D}'(\mathbb{R}^n)$ as $\epsilon \to 0$. But be careful, this does not imply convergence in $W^{m,p}(V)$, see Exercise 15.6. We need some more work, and use some specific property of ρ_ϵ.

Since, for every $\alpha \in \mathbb{N}^n$, we have $\mathrm{D}^\alpha u_\epsilon = (\mathrm{D}^\alpha u) * \rho_\epsilon$, we only need to show that, if $u \in L^p_{\mathrm{loc}}(\Omega)$, then $u_\epsilon \to u$ in $L^p(V)$ for every $V \Subset \Omega$. This is the content of Lemma 15.10. □

Lemma 15.10. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^p_{\mathrm{loc}}(\Omega)$. If $V \Subset \Omega$, then $\lim_{\epsilon\to 0} \|u_\epsilon - u\|_{L^p(V)} = 0$, where u_ϵ is defined as in* (139)*.*

Proof. Let $\bar\epsilon > 0$ be such that $B(V,\bar\epsilon) = \bigcup_{x\in V} B(x,\bar\epsilon) \Subset \Omega$. If $x \in V$ and $\epsilon \in (0,\bar\epsilon)$, then

$$\begin{aligned} u_\epsilon(x) &= \int_{B(0,\epsilon)} u(x-y)\rho_\epsilon(y)\,\mathrm{d}y \\ &= \int_{B(0,\epsilon)} u(x-y)\rho_\epsilon(y)^p \cdot \rho_\epsilon(y)^{1-1/p}\,\mathrm{d}y \\ &\overset{\text{(Hölder)}}{\le} \left(\int_{B(0,\epsilon)} |u(x-y)|^p \rho_\epsilon(y)\,\mathrm{d}y\right)^{1/p} \left(\int_{B(0,\epsilon)} \rho_\epsilon(y)\,\mathrm{d}y\right)^{1/p'} \\ &= \left(\int_{B(0,\epsilon)} |u(x-y)|^p \rho_\epsilon(y)\,\mathrm{d}y\right)^{1/p}. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_V |u_\epsilon(x)|^p\,\mathrm{d}x &\le \int_V \int_{B(0,\epsilon)} |u(x-y)|^p \rho_\epsilon(y)\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{B(0,\epsilon)} \int_V |u(x-y)|^p \rho_\epsilon(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\le \int_{B(V,\epsilon)} |u(x)|^p\,\mathrm{d}x. \end{aligned} \tag{140}$$

Next, fix $\eta > 0$. By Theorem 3.1, there exists $g \in C(\Omega)$ such that $\|g - u\|_{L^p(B(V,\bar\epsilon))} < \eta$. For $\epsilon \in (0,\bar\epsilon)$, we have

$$\begin{aligned} \|u - u_\epsilon\|_{L^p(V)} &\le \|u - g\|_{L^p(V)} + \|g - g_\epsilon\|_{L^p(V)} + \|g_\epsilon - u_\epsilon\|_{L^p(V)} \\ &\overset{(140)}{\le} \|g - u\|_{L^p(B(V,\bar\epsilon))} + \|g - g_\epsilon\|_{L^\infty(V)} \mathscr{L}^n(V) + \|g - u\|_{L^p(B(V,\bar\epsilon))} \\ &\le 2\eta + \|g - g_\epsilon\|_{L^\infty(V)} \mathscr{L}^n(V), \end{aligned}$$

Since g is continuous and V is compact, we know that $\|g - g_\epsilon\|_{L^\infty(V)} \to 0$ as $\epsilon \to 0$. Therefore, there is $\bar{\bar\epsilon}$ such that, if $\epsilon \in (0,\bar{\bar\epsilon})$, then $\|u - u_\epsilon\|_{L^p(V)} < 3\eta$. □

§15.5. Smooth approximation of Sobolev functions: interior approximation.

Lemma 15.11. *Let $\Omega \subset \mathbb{R}^n$ be an open set. Then there are a countable set $\mathscr{N} \subset \Omega$ and $r : \mathscr{N} \to (0,\infty)$ such that*

(15.11.a) for every $x \in \mathscr{N}$, $B(x, 2r_x) \subset \Omega$,
(15.11.b) $\Omega \subset \bigcup_{x\in\mathscr{N}} B(x, r_x)$,
(15.11.c) for every $y \in \Omega$, $\#\{x \in \mathscr{N} : y \in B(x, 2r_x)\} \le 2^{5n} < \infty$.

Moreover, are also functions $\{\phi_x\}_{x\in\mathscr{N}}$ such that, for each $x \in \mathscr{N}$, $\phi_x \in C^\infty_c(B(x, r_x))$, $0 \le \phi_x \le 1$, and such that $\sum_{x\in\mathscr{N}} \phi_x(y) = 1$ for every $y \in \Omega$.

Proof. If $\Omega = \mathbb{R}^n$, the proof is left as an exercise. Suppose that $\Omega \neq \mathbb{R}^n$, that is, $\partial\Omega \neq \emptyset$. Define $\delta : \Omega \to (0,+\infty)$, $\delta(x) = \mathrm{dist}(x, \partial\Omega)$, and, for $k \in \mathbb{Z}$,

$$\Omega_k = \{x \in \Omega : 2^k \le \delta(x) < 2^{k+1}\}.$$

For each $k \in \mathbb{Z}$, let $\mathscr{N}_k \subset \Omega_k$ be a maximal 2^{k-3}-separated set, that is, a subset of Ω_k such that, if $x, y \in \mathscr{N}_k$, then $B(x, 2^{k-3}) \cap B(y, 2^{k-3}) = \emptyset$, and for every $z \in \Omega_k$ there is $x \in \mathscr{N}_k$ with $|z - x| < 2 \cdot 2^{k-3} = 2^{k-2}$. The sets $\mathscr{N}_k$ are countable, thus $\mathscr{N} = \bigcup_{k\in\mathbb{Z}} \mathscr{N}_k$ is also countable. We claim that the set of points $\mathscr{N}$ with the radii $r_x = 2^{k-2}$ for $x \in \mathscr{N}_k$ is the wanted set.

The requirement (15.11.a) is clear. It is also clear that

$$\Omega \subset \bigcup_{x\in\mathscr{N}} B(x, r_x) = \bigcup_{k\in\mathbb{Z}} \bigcup_{x\in\mathscr{N}_k} B(x, 2^{k-2}),$$

and thus (15.11.b).

We need to show the upper bound in (15.11.c). Let $y \in \Omega$ and set $A(y) = \{x \in \mathscr{N} : y \in B(x, 2r_x)\}$. Keep in mind that, if $x \in \mathscr{N}_k$, then $r_x = 2^{k-2}$ and $2r_x = 2^{k-1}$. Let $j \in \mathbb{Z}$ such that $y \in \Omega_j$.

Notice that

$$x \in \mathscr{N}_k \quad \Rightarrow \quad B(x, 2^{k-1}) \subset \Omega_{k-1} \cup \Omega_k \cup \Omega_{k+1}. \tag{141}$$

Indeed, if $z \in B(x, 2^{k-1})$, then

$$\begin{aligned} \delta(z) &\le \delta(x) + 2^{k-1} < 2^{k+1} + 2^{k-1} \le 2^{k+2}, \\ \delta(z) &\ge \delta(x) - 2^{k-1} \ge 2^k - 2^{k-1} = 2^{k-1}. \end{aligned}$$

From (141) we get that, if $x \in \mathscr{N}_k$ and $y \in B(x, 2^{k-1})$, then $k \in \{j-1, j, j+1\}$. It follows that $A(y) \subset \mathscr{N}_{j-1} \cup \mathscr{N}_j \cup \mathscr{N}_{j+1}$.

We thus have that $A(y) \subset B(y, 2^{(j+1)-1}) = B(y, 2^j)$, and that balls of radius 2^{j-4} centered at points of $A(y)$ are pairwise disjoint. By Exercise 15.12, we conclude that $\#A(y) \le 2^{5n}$.

The construction of the functions ϕ_x is left as an exercise. □

Exercise 15.12. Let $j \in \mathbb{Z}$. Show that, if $A \subset B(0, 2^j)$ is such that, for every $a_1, a_2 \in A$ distinct, $B(a_1, 2^{j-4}) \cap B(a_2, 2^{j-4}) = \emptyset$, then $\#A \le 2^{5n}$.

Hint: The union of the pairwise disjoint balls $\{B(a, 2^{j-4})\}_{a \in A}$ is a subset of $B(0, 2^j)$, so its volume... ♢

Exercise 15.13. Construct the functions ϕ_x in Lemma 15.11. ♢

Theorem 15.14. *Let $\Omega \subset \mathbb{R}^n$ open. If $u \in W^{k,p}(\Omega)$, then there is a sequence $\{u_j\}_{j \in \mathbb{N}} \subset W^{k,p}(\Omega) \cap C^\infty(\Omega)$ such that $u_j \to u$ in $W^{k,p}(\Omega)$.*

Proof. We use the following notation: if B is the ball $B(x, r)$, then $2B$ is the ball $B(x, 2r)$.

From Lemma 15.11, we get a partition of unity $\{\phi_\ell\}_{\ell \in \mathbb{N}}$ and a countable family of balls $\{B_\ell\}_{\ell \in \mathbb{N}}$ such that $\Omega = \bigcup_{\ell \in \mathbb{N}} B_\ell$, $\mathrm{spt}(\phi_\ell) \subset B_\ell$, and $\#\{\ell : y \in 2B_\ell\} < \infty$ for all $y \in \Omega$.

By Theorem 15.9, for every $\ell \in \mathbb{N}$ and $k \in \mathbb{N}$, there is $\epsilon_k > 0$ such that the function

$$u_{k,\ell} = (\phi_\ell u) * \rho_{\epsilon_k}$$

is supported in $2B_\ell$ and $\|u_{k,\ell} - \phi_\ell u\|_{W^{m,p}(2B_\ell)} < \frac{1/k}{2^\ell}$. Define

$$u_k = \sum_{\ell \in \mathbb{N}} u_{k,\ell}.$$

Notice that $u_k \in C^\infty(\Omega)$, because each $u_{k,\ell}$ is smooth and the sum is locally finite. Moreover,

$$\begin{aligned} \|u_k - u\|_{W^{m,p}(\Omega)} &\le \sum_{\ell \in \mathbb{N}} \|u_{k,\ell} - \phi_\ell u\|_{W^{m,p}(\Omega)} \\ [\text{because } \mathrm{spt}(u_{k,\ell} - \phi_\ell u) \subset 2B_\ell] &= \sum_{\ell \in \mathbb{N}} \|u_{k,\ell} - \phi_\ell u\|_{W^{m,p}(2B_\ell)} \\ &\le \sum_{\ell \in \mathbb{N}} \frac{1/k}{2^\ell} = \frac{2}{k}. \end{aligned}$$

Therefore, $u_k \to u$ in $W^{m,p}(\Omega)$. □

§15.6. Smooth approximation of Sobolev functions: global approximation. For a proof of the following theorem, see [5, §5.3.3].

Theorem 15.15. *Let $\Omega \subset \mathbb{R}^n$ open with ∂U of class C^1. If $u \in W^{k,p}(\Omega)$, then there is a sequence $\{u_j\}_{j \in \mathbb{N}} \subset W^{k,p}(\Omega) \cap C^\infty(\bar{\Omega})$ such that $u_j \to u$ in $W^{k,p}(\Omega)$.*

§15.7. The closure of smooth functions with compact support. Using Theorem 3.1 and some smoothing, one can show that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$, see Exercise 15.17 However, it is NOT true in general that $C_c^\infty(\Omega)$ is dense in $W^{m,p}(\Omega)$.

Exercise 15.16. Show that, for every $p \in [1, \infty)$, the constant function $u \equiv 1$ belongs to $W^{1,p}((0,1))$, but it is not the limit in $W^{1,p}$ of functions $C_c^\infty((0,1))$. ◊

So, for $\Omega \subset \mathbb{R}^n$ open, $m \in \mathbb{N}$ and $p \in [1, +\infty]$, we define

$$W_0^{m,p}(\Omega) = \{u \in W^{m,p}(\Omega) : \exists \{u_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\Omega) \text{ such that } u_j \to u \text{ in } W^{m,p}(\Omega)\}.$$

Exercise 15.17. Show that $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$. ◊

Theorem 15.18. *For every $m \in \mathbb{N}$ and $p \in [1, \infty)$, $W_0^{m,p}(\mathbb{R}^n) = W^{m,p}(\mathbb{R}^n)$.*

Proof. Given $u \in W^{m,p}(\mathbb{R}^n)$, we need to find a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$ that converges to u in $W^{m,p}(\mathbb{R}^n)$. By Theorem 15.14 and a diagonal argument, we can assume $u \in C^\infty(\mathbb{R}^n)$.

Let $\zeta : \mathbb{R}^n \to [0, 1]$ be a smooth function such that

$$B(0, 1) \subset \{\zeta = 1\} \subset \operatorname{spt}(\zeta) \subset \bar{B}(0, 2).$$

For $R > 0$ and $x \in \mathbb{R}^n$, define $\zeta_R(x) = \zeta(x/R)$. Notice that, for every $R > 0$, $\beta \in \mathbb{N}^n$ and $x \in \mathbb{R}^n$,

$$|\mathrm{D}^\beta \zeta_R(x)| = |R^{-|\beta|}\mathrm{D}^\beta \zeta(x/R)| \le \frac{1}{R^{|\alpha|}} \|\mathrm{D}^\beta \zeta\|_{L^\infty(\mathbb{R}^n)}.$$

We shall approximate u with $\zeta_R u$ as $R \to \infty$. Notice that, for $R > 0$ and $\alpha \in \mathbb{N}^n$, we have

$$\begin{aligned}\|\mathrm{D}^\alpha u - \mathrm{D}^\alpha(\zeta_R u)\|_{L^p(\mathbb{R}^n)} &\le \|\mathrm{D}^\alpha u - \zeta_R \mathrm{D}^\alpha u\|_{L^p(\mathbb{R}^n)} + \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \|\mathrm{D}^\beta \zeta_R \cdot \mathrm{D}^{\alpha-\beta} u\|_{L^p(\mathbb{R}^n)} \\ &\le \left(\int_{\mathbb{R}^n \setminus B(0,R)} |\mathrm{D}^\alpha u(x)|^p \,\mathrm{d}x\right)^{1/p} + \sum_{\beta\le\alpha} \binom{\alpha}{\beta} \frac{1}{R^{|\beta|}} \|\mathrm{D}^\alpha \zeta\|_{L^\infty(\mathbb{R}^n)} \|\mathrm{D}^{\alpha-\beta} u\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Since

$$\lim_{R\to\infty} \int_{\mathbb{R}^n \setminus B(0,R)} |\mathrm{D}^\alpha u(x)|^p \,\mathrm{d}x = 0,$$

we conclude that $\lim_{R\to\infty} \|u - \zeta_R u\|_{W^{m,p}(\mathbb{R}^n)} = 0$. □

§15.8. Sobolev inequalities: $1 \le p < n$.

Theorem 15.19. *Let $n \ge 2$. If $u \in C_c^1(\mathbb{R}^n)$,*

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{n}{n-1}} \,\mathrm{d}x\right)^{\frac{n-1}{n}} \le \int_{\mathbb{R}^n} |\nabla u(x)| \,\mathrm{d}x. \tag{142}$$

Proof. Let $u \in C_c^1(\mathbb{R}^n)$. For $j \in \{1, \dots, n\}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$ we denote by $u(x|y)$ the evaluation $u(x|_j y) = u(x_1, \dots, x_{j-1}, y, x_{j+1}, \dots, x_n)$. Notice that, for every $x \in \mathbb{R}^n$ and $j \in \{1, \dots, n\}$,

$$u(x) = \int_{-\infty}^{x_j} \partial_j u(x|_j y_j) \,\mathrm{d}y_j.$$

Therefore, for every $x \in \mathbb{R}^n$,

$$|u(x)|^{\frac{n}{n-1}} \le \prod_{j=1}^n \left(\int_{-\infty}^{\infty} |\partial_j u(x|_j y_j)| \,\mathrm{d}y_j\right)^{\frac{1}{n-1}} \le \prod_{j=1}^n \left(\int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j\right)^{\frac{1}{n-1}}. \tag{143}$$

We claim that, for each $\ell \in \{1, \dots, n\}$, we have

$$\int_{\mathbb{R}^\ell} |u(x)|^{\frac{n}{n-1}} \,\mathrm{d}x_{\le\ell} \le \left(\int_{\mathbb{R}^\ell} |\nabla u(x)| \,\mathrm{d}x_{\le\ell}\right)^{\frac{\ell}{n-1}} \prod_{j=\ell+1}^n \left(\int_{\mathbb{R}^\ell} \int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j \,\mathrm{d}x_{\le\ell}\right)^{\frac{1}{n-1}}. \tag{144}$$

where $\mathrm{d}x_{\le\ell} = \mathrm{d}x_1 \cdots \mathrm{d}x_\ell$. To prove (144) for $\ell = 1$, we integrate (143) in x_1:

$$\begin{aligned}
&\int_{\mathbb{R}} |u(x)|^{\frac{n}{n-1}} \,\mathrm{d}x_1 \\
&\overset{(143)}{\le} \int_{\mathbb{R}} \prod_{j=1}^n \left(\int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&= \left(\int_{\mathbb{R}} |\nabla u(x|_1 y_1)| \,\mathrm{d}y_1 \right)^{\frac{1}{n-1}} \int_{\mathbb{R}} \prod_{j=2}^n \left(\int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j \right)^{\frac{1}{n-1}} \mathrm{d}x_1 \\
&\overset{\text{(General Hölder)}}{\le} \left(\int_{\mathbb{R}} |\nabla u(x)| \,\mathrm{d}x_1 \right)^{\frac{1}{n-1}} \prod_{j=2}^n \left(\int_{\mathbb{R}} \int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j \,\mathrm{d}x_1 \right)^{\frac{1}{n-1}} .
\end{aligned}$$

Next, given (144) for $\ell = k \in \{1, \dots, n-1\}$, we prove the same for $\ell = k+1$ integrating in x_{k+1}:

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{\mathbb{R}^k} |u(x)|^{\frac{n}{n-1}} \,\mathrm{d}x_{\le k} \,\mathrm{d}x_{k+1} \\
&\overset{(144)}{\le} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^k} |\nabla u(x)| \,\mathrm{d}x_{\le k} \right)^{\frac{k}{n-1}} \prod_{j=k+1}^n \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j \,\mathrm{d}x_{\le k} \right)^{\frac{1}{n-1}} \mathrm{d}x_{k+1} \\
&= \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}} |\nabla u(x|_{k+1} y_{k+1})| \,\mathrm{d}y_{k+1} \,\mathrm{d}x_{\le k} \right)^{\frac{1}{n-1}} \times \\
&\quad \times \int_{\mathbb{R}} \left(\int_{\mathbb{R}^k} |\nabla u(x)| \,\mathrm{d}x_{\le k} \right)^{\frac{k}{n-1}} \prod_{j=k+2}^n \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j \,\mathrm{d}x_{\le k} \right)^{\frac{1}{n-1}} \mathrm{d}x_{k+1} \\
&\overset{\text{(General Hölder)}}{\le} \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}} |\nabla u(x|_{k+1} y_{k+1})| \,\mathrm{d}y_{k+1} \,\mathrm{d}x_{\le k} \right)^{\frac{1}{n-1}} \times \\
&\quad \times \left(\int_{\mathbb{R}} \int_{\mathbb{R}^k} |\nabla u(x)| \,\mathrm{d}x_{\le k} \,\mathrm{d}x_{k+1} \right)^{\frac{k}{n-1}} \prod_{j=k+2}^n \left(\int_{\mathbb{R}} \int_{\mathbb{R}^k} \int_{\mathbb{R}} |\nabla u(x|_j y_j)| \,\mathrm{d}y_j \,\mathrm{d}x_{\le k} \,\mathrm{d}x_{k+1} \right)^{\frac{1}{n-1}} ,
\end{aligned}$$

which gives (144) for $\ell = k+1$.

Finally, notice that (144) for $\ell = n$ is (142). □

Corollary 15.20. *Let $n \ge 2$. Then* (142) *holds for all $u \in W^{1,1}(\mathbb{R}^n)$.*

Proof. Let $u \in W^{1,1}(\mathbb{R}^n)$. By Theorem 15.18, there is a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C^1_c(\mathbb{R}^n)$ such that $u_j \to u$ in $W^{1,1}(\mathbb{R}^n)$. Notice that,

$$\|u_j - u_k\|_{L^{\frac{n}{n-1}}} \overset{(142)}{\le} \|\nabla(u_j - u_k)\|_{L^1} \le \|u_j - u_k\|_{W^{1,1}}.$$

Therefore, $\{u_j\}_j$ is a Cauchy sequence in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$. Since $u_j \to u$ in L^1 already, then u_j must converge to u in $L^{\frac{n}{n-1}}(\mathbb{R}^n)$ too. We conclude that

$$\|u\|_{L^{\frac{n}{n-1}}} = \lim_{j\to\infty} \|u_j\|_{L^{\frac{n}{n-1}}} \overset{(142)}{\le} \lim_{j\to\infty} \|\nabla u_j\|_{L^1} = \|\nabla u\|_{L^1}.$$

□

Exercise 15.21. In Theorem 15.19 we required $n \ge 2$. What happens when $n = 1$? ◊

§15.9. **A remark on the isoperimetric inequality.** The inequality (142) has a geometric meaning: it is indeed equivalent to the *isoperimetric inequality*. More precisely, the characteristic function $\mathbb{1}_E$ of a measurable $E \subset \mathbb{R}^n$ belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and thus it defines a distribution: $\mathbb{1}_E \in \mathscr{D}'(\mathbb{R}^n)$. Define the *perimeter* of E as the *total variation* of $\mathrm{D}E$, that is,

$$\mathrm{Per}(E) = \sup\{\sum_{j=1}^n \partial_j \mathbb{1}_E[\phi_j] : \phi_j \in \mathscr{D},\ \|\phi_j\|_{L^\infty} \le 1\}.$$

In order to put this formula into perspective, we define for $u \in \mathscr{D}'$

$$|\mathrm{D}u|(\mathbb{R}^n) = \sup\{\sum_{j=1}^n \partial_j u[\phi_j] : \phi_j \in \mathscr{D},\ \|\phi_j\|_{L^\infty} \leq 1\} \in [0, +\infty].$$

It is clear that $|\mathrm{D}u|(\mathbb{R}^n) < \infty$ if and only if $|\mathrm{D}u|$ is a Radon measure, as seen in Proposition 13.12. In this case, we say that u has *bounded variation*. The space of *functions with bounded variation*, *BV functions* for short, is

$$\mathrm{BV}(\mathbb{R}^n) = \{u \in L^1(\mathbb{R}^n) : |\mathrm{D}u|(\mathbb{R}^n) < \infty\}.$$

If $u \in C_c^\infty(\mathbb{R}^n)$, then $|\mathrm{D}u|(\mathbb{R}^n) = \int_{\mathbb{R}^n} |\nabla u| \,\mathrm{d}x$. One can show that (142) extends to BV functions: if $u \in \mathrm{BV}(\mathbb{R}^n)$, then

$$\|u\|_{L^{\frac{n}{n-1}}} \leq |\mathrm{D}u|(\mathbb{R}^n). \tag{145}$$

The proof is by smooth approximation, similarly to the proof of Corollary 15.20. If we apply (145) to $\mathbb{1}_E$, we get

$$|E|^{\frac{n-1}{n}} \leq \mathrm{Per}(E), \tag{146}$$

where $|E|$ is the volume of E. This is the isoperimetric inequality in $\mathbb{R}^n$. Equality in (146) is attained only when E is an Euclidean ball.

In fact, one can show that from (146), using the coarea inequality one can get (142) for $u \in C_c^1(\mathbb{R}^n)$.

§15.10. Gagliardo–Nirenberg–Sobolev inequality. The *Sobolev conjugate* of $p \in [1, n)$ is the number p^* such that

$$\frac{1}{p} - \frac{1}{p^*} = \frac{1}{n}, \text{ or, equivalently, } \frac{1}{p^*} = \frac{1}{p} - \frac{1}{n}.$$

In other words,

$$p^* = \frac{np}{n-p}.$$

Notice that, (142) is (147) for $p = 1$.

Theorem 15.22 (Gagliardo–Nirenberg–Sobolev inequality). *Let $n \geq 2$ and $p \in [1, n)$ with Sobolev conjugate $p^* = \frac{np}{n-p}$. There exists $C \in \mathbb{R}$ such that, for all $u \in C_c^1(\mathbb{R}^n)$,*

$$\|u\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\nabla u\|_{L^p(\mathbb{R}^n)}. \tag{147}$$

In fact, we can take $C = \frac{p(n-1)}{n-p} = \frac{p^*}{1^*}$.

Proof. If $p = 1$, then (147) is (142). Therefore, we assume $p > 1$.

Fix $u \in C_c^1(\mathbb{R}^n)$. Let $\gamma > 1$ and set $v = |u|^\gamma$. By Exercise 15.23, $v \in C_c^1(\mathbb{R}^n)$. Therefore, by Corollary 15.20, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} |u|^{\gamma\frac{n}{n-1}} \,\mathrm{d}x\right)^{\frac{n-1}{n}} &= \left(\int_{\mathbb{R}^n} v^{\frac{n}{n-1}} \,\mathrm{d}x\right)^{\frac{n-1}{n}} \\
&\overset{(142)}{\leq} \int_{\mathbb{R}^n} |\nabla v(x)| \,\mathrm{d}x \\
&= \gamma \int_{\mathbb{R}^n} |u|^{\gamma-1} |\nabla u(x)| \,\mathrm{d}x \\
&\overset{(\text{Hölder})}{\leq} \gamma \left(\int_{\mathbb{R}^n} |u|^{\frac{p}{p-1}(\gamma-1)} \,\mathrm{d}x\right)^{\frac{p-1}{p}} \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \,\mathrm{d}x\right)^{1/p}.
\end{aligned}$$

We want $\gamma \frac{n}{n-1} = \frac{p}{p-1}(\gamma - 1)$, that is $\gamma = \frac{p(n-1)}{n-p}$. Notice that, if $p \in (1, n)$, then $\gamma > 1$. If $u \equiv 0$, then (147) trivially holds. So, we can assume $u \neq 0$, and thus the above estimates, with this γ, gives

$$\left(\int_{\mathbb{R}^n} |u|^{\gamma\frac{n}{n-1}} \,\mathrm{d}x\right)^{\frac{n-1}{n} - \frac{p-1}{p}} \leq \gamma \left(\int_{\mathbb{R}^n} |\nabla u(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

A direct computations gives $\gamma \frac{n}{n-1} = p^*$ and $\frac{n-1}{n} - \frac{p-1}{p} = \frac{1}{p^*}$. So, we have (147). □

Exercise 15.23. Show that, if $u \in C^1_c(\mathbb{R}^n)$ and $\gamma > 1$, then $|u|^\gamma \in C^1_c(\mathbb{R}^n)$ and $\nabla(|u|^\gamma) = \gamma|u|^{\gamma-1}\nabla u$.

Hint: on the set $\{u \neq 0\}$, there the statement is trivial. What remains to show is that, if $u(x) = 0$, then $|u|^\gamma$ is differentiable at x with derivative equal to 0. $\lozenge$

Corollary 15.24. *Let $n \geq 2$ and $p \in [1, n)$ with Sobolev conjugate $p^* = \frac{np}{n-p}$. Then* (147) *holds for all $u \in W^{1,p}(\mathbb{R}^n)$.*

Proof. Let $u \in W^{1,p}(\mathbb{R}^n)$. By Theorem 15.18, there is a sequence $\{u_j\}_{j\in\mathbb{N}} \subset C^1_c(\mathbb{R}^n)$ such that $u_j \to u$ in $W^{1,p}(\mathbb{R}^n)$. Notice that,

$$\|u_j - u_k\|_{L^{p^*}} \overset{(147)}{\leq} \|\nabla(u_j - u_k)\|_{L^p} \leq \|u_j - u_k\|_{W^{1,p}}.$$

Therefore, $\{u_j\}_j$ is a Cauchy sequence in $L^{p^*}(\mathbb{R}^n)$. Since $u_j \to u$ in L^1 already, then u_j must converge to u in $L^{p^*}(\mathbb{R}^n)$ too. We conclude that

$$\|u\|_{L^{p^*}} = \lim_{j\to\infty} \|u_j\|_{L^{p^*}} \overset{(147)}{\leq} \lim_{j\to\infty} \|\nabla u_j\|_{L^p} = \|\nabla u\|_{L^p}.$$

□

§15.11. Sobolev inequalities: $n < p < \infty$. Morrey's inequality.

Theorem 15.25 (Morrey's inequality)**.** *Let $n \geq 2$ and $p \in (n, \infty)$. There is $C \in \mathbb{R}^n$ such that, for every $o \in \mathbb{R}$, $R > 0$ and $u \in W^{1,p}(B(o, 4R))$, and for two every Lebesgue points $x, y \in B(o, R)$ of u (thus, for a.e. $x, y \in B(o, R)$),*

$$|u(x) - u(y)| \leq C|x - y|^{1-n/p}\|\nabla u\|_{L^p(B(o,4R))}. \tag{148}$$

In particular, if $\Omega \subset \mathbb{R}^n$ is open, then every $u \in W^{1,p}_{\text{loc}}(\Omega)$ has a locally $(1 - n/p)$-Hölder continuous representative.

The constant C, which depends on p and n, can be taken to be $C = \frac{2^n (n\omega_n)^{\frac{p-1}{p}}}{\omega_{n-1}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}}$.

We shall prove Theorem 15.25 after a few intermediate statements.

Proposition 15.26. *For every $n \geq 2$ there is $C \in \mathbb{R}$ such that, for every $u \in C^1(\mathbb{R}^n)$, every $x \in \mathbb{R}^n$ and every $r > 0$,*

$$\fint_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y \leq C \int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{n-1}} \,\mathrm{d}y. \tag{149}$$

The constant C can be taken equal to $C = \frac{1}{\omega_n n}$.

Proof.

$$\begin{aligned}
\int_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y &= \int_0^r \int_{\partial B(x,s)} |u(y) - u(x)| \,\mathrm{d}S(y) \,\mathrm{d}s \\
&= \int_0^r \int_{\partial B(0,1)} |u(x + sz) - u(x)| s^{n-1} \,\mathrm{d}S(z) \,\mathrm{d}s \\
&= \int_0^r \int_{\partial B(0,1)} \left| \int_0^s \nabla u(x + sz) \cdot z \,\mathrm{d}t \right| s^{n-1} \,\mathrm{d}S(z) \,\mathrm{d}s \\
&\leq \int_0^r \int_{\partial B(0,1)} \int_0^s |\nabla u|(x + sz) \,\mathrm{d}t s^{n-1} \,\mathrm{d}S(z) \,\mathrm{d}s \\
&= \int_0^r \int_0^s \int_{\partial B(0,1)} \frac{|\nabla u|(x + sz)}{t^{n-1}} t^{n-1} \,\mathrm{d}S(z) \,\mathrm{d}t s^{n-1} \,\mathrm{d}s \\
&= \int_0^r \int_{B(x,s)} \frac{|\nabla u|(y)}{|y - x|^{n-1}} \,\mathrm{d}y s^{n-1} \,\mathrm{d}s \\
&\leq \int_0^r \int_{B(x,r)} \frac{|\nabla u|(y)}{|y - x|^{n-1}} \,\mathrm{d}y s^{n-1} \,\mathrm{d}s \\
&= \frac{r^n}{n} \int_{B(x,r)} \frac{|\nabla u|(y)}{|y - x|^{n-1}} \,\mathrm{d}y.
\end{aligned}$$

Since the volume of $B(x, r)$ is $\omega_n r^n$, then we obtain (149) with $C = \frac{1}{\omega_n n}$. □

Exercise 15.27. Compute left and right hand sides of (149) for $u(y) = |y|$. ♢

Remark 15.28. Proposition 15.26 is important because the integral kernel in the right-hand side of (149) is the famous *Riesz potential.*

Exercise 15.29. Show that, for every $n \geq 2$, for every $u \in C^1(\mathbb{R}^n)$, every $x \in \mathbb{R}^n$ and every $r > 0$,

$$\left| \fint_{B(x,r)} u(y)\,\mathrm{d}y - u(x) \right| \leq \fint_{B(x,r)} |u(y) - u(x)|\,\mathrm{d}y.$$

Solution. Since $\fint_{B(x,r)} \mathrm{d}y = 1$, we have

$$\left| \fint_{B(x,r)} u(y)\,\mathrm{d}y - u(x) \right| = \left| \fint_{B(x,r)} (u(y) - u(x))\,\mathrm{d}y \right| \leq \fint_{B(x,r)} |u(y) - u(x)|\,\mathrm{d}y.$$

♢

Proposition 15.30. *For every $n \geq 2$ and $p \in (n, +\infty)$, there is $C(p,n) \in \mathbb{R}$ such that, for every $u \in C^1(\mathbb{R}^n)$, every $x \in \mathbb{R}^n$ and every $r > 0$,*

$$\fint_{B(x,r)} |u(y) - u(x)|\,\mathrm{d}y \leq C(p,n) r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B(x,r))}. \tag{150}$$

The constant $C(p,n)$ can be taken equal to $C(p,n) = (n\omega_n)^{-\frac{1}{p}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}}$.

Proof. We simply apply the Hölder inequality to (149):

$$\begin{aligned}
\fint_{B(x,r)} |u(y) - u(x)|\,\mathrm{d}y &\overset{(149)}{\leq} \frac{1}{\omega_n n} \int_{B(x,r)} \frac{|\nabla u(y)|}{|y - x|^{n-1}}\,\mathrm{d}y \\
&\overset{(\text{Hölder})}{\leq} \frac{1}{\omega_n n} \left(\int_{B(x,r)} |\nabla u(y)|^p\,\mathrm{d}y \right)^{\frac{1}{p}} \left(\int_{B(x,r)} \frac{1}{|y - x|^{(n-1)\frac{p}{p-1}}}\,\mathrm{d}y \right)^{\frac{p-1}{p}}.
\end{aligned}$$

Notice that, using polar coordinates,

$$\begin{aligned}
\int_{B(x,r)} \frac{1}{|y - x|^{(n-1)\frac{p}{p-1}}}\,\mathrm{d}y &= \int_0^r n\omega_n s^{n-1} \frac{1}{s^{(n-1)\frac{p}{p-1}}}\,\mathrm{d}s \\
&= n\omega_n \int_0^r s^{-\frac{n-1}{p-1}}\,\mathrm{d}s \\
&= n\omega_n \frac{p-1}{p-n} r^{\frac{p-n}{p-1}},
\end{aligned}$$

where $p > n$ ensures that $-\frac{n-1}{p-1} > -1$ and thus that the above integral is finite. Therefore,

$$\left(\int_{B(x,r)} \frac{1}{|y - x|^{(n-1)\frac{p}{p-1}}}\,\mathrm{d}y \right)^{\frac{p-1}{p}} = \left(n\omega_n \frac{p-1}{p-n} \right)^{\frac{p-1}{p}} r^{\frac{p-n}{p}},$$

from which we conclude (149) with $C(p,n) = (n\omega_n)^{\frac{p-1}{p}-1} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}} = (n\omega_n)^{-\frac{1}{p}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}}$.
Finally, since We have obtained □

Remark 15.31. For $W \subset \mathbb{R}^n$ bounded and with positive volume, and $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, define

$$u_W = \fint_W u(y)\,\mathrm{d}y = \frac{1}{|W|} \int_W u(y)\,\mathrm{d}y,$$

where $|W| = \mathscr{L}^n(W)$ is the volume of W. In the proof of Proposition 15.33, we will use an idea that is worth to keep in mind and understand correctly. Suppose we have two points $x, z \in \mathbb{R}^n$ with $r = |x - z|$, and set $W = B(x,r) \cap B(z,r)$. Clearly we have $|u(x) - u(z)| \leq |u(x) - u_W| + |u_W - u(z)| \leq \fint_W |u(y) - u(x)|\,\mathrm{d}y + \fint_W |u(y) - u(z)|\,\mathrm{d}y$. We now do the following:

$$\fint_W |u(y) - u(x)|\,\mathrm{d}y = \frac{|B(x,r)|}{|W|} \frac{1}{|B(x,r)|} \int_W |u(y) - u(x)|\,\mathrm{d}y$$

$$\leq \frac{|B(x,r)|}{|W|} \frac{1}{|B(x,r)|} \int_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y$$
$$= \frac{|B(x,r)|}{|W|} \fint_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y.$$

Now we have that $\frac{|B(x,r)|}{|W|} = \frac{|B(x,r)|}{|B(x,r)\cap B(z,r)|}$ is a constant that depends only on n. Apart of computing it explicitely, see Exercise 15.32, we can convince ourselves that it is a constant because, clearly, for every $v \in \mathbb{R}^n$, $\lambda > 0$ and $O \in \mathsf{O}(n)$,

$$\frac{|B(x+v,r)|}{|B(x+v,r) \cap B(z+v,r)|} = \frac{|B(x,\lambda r)|}{|B(x,\lambda r) \cap B(z,\lambda r)|} = \frac{|B(Ox,r)|}{|B(Ox,r) \cap B(Oz,r)|}.$$

So, $\frac{|B(x,r)|}{|B(x,r)\cap B(z,r)|} = \frac{|B(0,1)|}{|B(0,1)\cap B(e_1,1)|}$.

Exercise 15.32. Compute $\frac{|B(x,r)|}{|B(x,r)\cap B(z,r)|}$.

Solution. First check that $B(0,1) \cap B(e_1,1) = \{(y_1,y') \in \mathbb{R}^n : y_1 \in (0,1),\ y' \in \mathbb{R}^{n-1},\ |y'|^2 \leq y_1^2,\ |y'|^2 \leq 1 - y_1^2\}$. Then check $|B(0,1)\cap B(e_1,1)| = 2\int_0^{1/2} \omega_{n-1} y_1^{n-1} \,\mathrm{d}y_1 = \frac{\omega_{n-1}}{n2^{n-1}}$. Finally, $\frac{|B(x,r)|}{|B(x,r)\cap B(z,r)|} = \frac{|B(0,1)|}{|B(0,1)\cap B(e_1,1)|} = \frac{n\omega_n 2^{n-1}}{\omega_{n-1}}$. ◊

Proposition 15.33 (Morrey's inequality for smooth functions)**.** *For every $n \geq 2$ and $p > n$, there is $C \in \mathbb{R}$ such that, for every $o \in \mathbb{R}$, $R > 0$ and $u \in C^1(B(o,3R))$,*

$$\forall x, y \in B(o,R) \qquad |u(x) - u(y)| \leq C|x-y|^{1-n/p} \|\nabla u\|_{L^p(B(o,3R))}. \tag{151}$$

The constant C, which depends on p and n, can be taken to be $C = \frac{2^n (n\omega_n)^{\frac{p-1}{p}}}{\omega_{n-1}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}}$.

Proof. Let $x, y \in B(o,R)$, and set $r = |x-y| \leq 2R$ and $W = B(x,r) \cap B(y,r)$. Using the argument described in Remark 15.31, with constant $C = \frac{|B(0,1)|}{|B(0,1)\cap B(e_1,1)|} = \frac{n\omega_n 2^{n-1}}{\omega_{n-1}}$ (see Exercise 15.32 for the explicit value of C),

$$\begin{aligned}
|u(x) - u(y)| &\leq |u(x) - u_W| + |u_W - u(y)| \\
&\leq \fint_W |u(z) - u(x)| \,\mathrm{d}z + \fint_W |u(z) - u(y)| \,\mathrm{d}z \\
&\leq C \fint_{B(x,r)} |u(z) - u(x)| \,\mathrm{d}z + C \fint_{B(y,r)} |u(z) - u(y)| \,\mathrm{d}z \\
&\overset{(150)}{\leq} CC(p,n) r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B(x,r))} + CC(p,n) r^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B(y,r))} \\
&\overset{(*)}{\leq} 2CC(p,n)|x-y|^{1-\frac{n}{p}} \|\nabla u\|_{L^p(B(o,3R))}.
\end{aligned}$$

In $(*)$ we have used that $B(x,r) \subset B(x,2R) \subset B(o,3R)$. Since $C(p,n) = (n\omega_n)^{-\frac{1}{p}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}}$,

$$2CC(p,n) = \frac{2^n (n\omega_n)^{\frac{p-1}{p}}}{\omega_{n-1}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}}.$$

□

Remark 15.34. The constant we have found is not necessarily optimal. Finding optimal constants in Sobolev inequalities may be quite tricky. See Remark 15.35 for another constant.

Remark 15.35. The proof presented here has been taken from Evans [5, §5.6.2]. Parviainen in [**Parviainen**] proves Proposition 15.33 with a slightly different approach. First, they don't go through the Riesz potential as in Proposition 15.26. Second, they use cubes instead of balls. Third, the constant they get in (151) is $C = \frac{2np}{p-n}$.

Exercise 15.36. For $n \geq 2$ and $p > n$, which of these constants is better in Morrey's inequality?

$$A = \frac{2^n (n\omega_n)^{\frac{p-1}{p}}}{\omega_{n-1}} \left(\frac{p-1}{p-n}\right)^{\frac{p-1}{p}}, \qquad B = \frac{2np}{p-n}.$$

◊

Exercise 15.37. Recall that a point $x \in \mathbb{R}^n$ is a *Lebesgue point* of $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ if

$$\lim_{r\to 0} \fint_{B(x,r)} |u(y) - u(x)| \,\mathrm{d}y = 0.$$

Recall also that, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then almost every $x \in \mathbb{R}^n$ is a Lebesgue point of u; see [8, Theorem (3.20), p.93].

Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a family of standard mollifiers on $\mathbb{R}^n$. For $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, set $u_\epsilon = u*\rho_\epsilon \in C^\infty(\mathbb{R}^n)$. Show that

$$\forall x \in \mathbb{R}^n \text{ Lebesgue point of } u, \qquad \lim_{\epsilon\to 0} u_\epsilon(x) = u(x). \tag{152}$$

Note that here we do not consider u "up to a set of measure zero", but really a fixed function $u : \mathbb{R}^n \to \mathbb{C}$.

Solution. Notice that, for every $y \in \mathbb{R}^n$, $|\rho_\epsilon(y)| = \left|\frac{\rho(y/\epsilon)}{\epsilon^n}\right| \le \frac{\|\rho\|_{L^\infty}}{\epsilon^n}$. Recall also that $\mathrm{spt}(\rho_\epsilon) \subset \bar{B}(0,\epsilon)$. Therefore,

$$\begin{aligned}
|u_\epsilon(x) - u(x)| &= \left| \int_{B(x,\epsilon)} u(x-y)\rho_\epsilon(y)\,\mathrm{d}y - u(x)\int_{B(x,\epsilon)} \rho_\epsilon(y)\,\mathrm{d}y \right| \\
&\le \int_{B(x,\epsilon)} |u(x-y) - u(x)|\rho_\epsilon(y)\,\mathrm{d}y \\
&\le \frac{\|\rho\|_{L^\infty}}{\epsilon^n} \int_{B(x,\epsilon)} |u(x-y) - u(x)|\,\mathrm{d}y \\
&= \|\rho\|_{L^\infty}\omega_n \fint_{B(x,\epsilon)} |u(x-y) - u(x)|\,\mathrm{d}y.
\end{aligned}$$

Therefore, if x is a Lebesgeue point of u, then $\lim_\epsilon u_\epsilon(x) = u(x)$.

See [8, Theorem 8.15] for a more general statement (for mollifications that do not have compact support). ♢

<u>*Proof of Theorem 15.25: Morrey's inequality for Sobolev functions.*</u> Let $u \in W^{1,p}(B(o,4R))$ and $x, y \in B(o,R)$ be two Lebesgue points of u.

Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a family of standard mollifiers on $\mathbb{R}^n$ and denote by $u_\epsilon = u * \rho_\epsilon \in C^\infty(\mathbb{R}^n)$. By Exercise 15.37, by the Morrey's inequality for smooth functions 15.33, and by standard properties of mollifiers, we have

$$\begin{aligned}
|u(x) - u(y)| &\overset{(152)}{=} \lim_{\epsilon\to 0} |u_\epsilon(x) - u_\epsilon(y)| \\
&\overset{(151)}{\le} \limsup_{\epsilon\to 0} C|x-y|^{1-n/p}\|\nabla u_\epsilon\|_{L^p(B(o,3R))} \\
&= C|x-y|^{1-n/p} \limsup_{\epsilon\to 0} \|(\nabla u)_\epsilon\|_{L^p(B(o,3R))} \\
&\overset{(110)}{\le} C|x-y|^{1-n/p}\|\nabla u\|_{L^p(B(o,4R))}.
\end{aligned}$$

In the last step we use Young's inequality (110) as follows: if $\epsilon < R$, then there is a cut-off function $\zeta \in C^\infty_c(B(o,4R)$ such that $\bar{B}(o, 3R+\epsilon) \subset \{\zeta = 1\}$ and $0 \le \zeta \le 1$. Then $(\zeta\nabla u)_\epsilon = (\nabla u)_\epsilon$ in $B(o,3R)$. So,

$$\begin{aligned}
\|(\nabla u)_\epsilon\|_{L^p(B(o,3R))} &= \|(\zeta\nabla u)_\epsilon\|_{L^p(B(o,3R))} \le \|(\zeta\nabla u)_\epsilon\|_{L^p(\mathbb{R}^n)} \\
&= \|\rho_\epsilon * (\zeta\nabla u)\|_{L^p(\mathbb{R}^n)} \le \|\rho_\epsilon * |\zeta\nabla u|\|_{L^p(\mathbb{R}^n)} \\
&\overset{(110)}{\le} \|\rho_\epsilon\|_{L^1(\mathbb{R}^n)}\|\zeta\nabla u\|_{L^p(\mathbb{R}^n)} \le \|\nabla u\|_{L^p(B(o,4R))}.
\end{aligned}$$

□

Exercise 15.38. Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ open. Show that, if $u \in W^{m,p}_{\mathrm{loc}}(\Omega)$ for some $m \ge 1$ and $p > n$, then $u \in C^{m-1}(\Omega)$. In particular, show that, for every $p > n$,

$$\bigcap_{m\ge 1} W^{m,p}_{\mathrm{loc}}(\Omega) = C^\infty(\Omega).$$

♢

§15.12. Difference quotients. Let $\Omega \subset \mathbb{R}^n$ open. For $\epsilon > 0$, define

$$\Omega_\epsilon = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > \epsilon\} = \{x \in \Omega : \bar{B}(x,\epsilon) \subset \Omega\}.$$

For $u : \Omega \to \mathbb{C}$, $j \in \{1, \dots, n\}$ and $h \neq 0$, define $\Delta_j^h u : \Omega_{|h|} \to \mathbb{C}$ as

$$\Delta_j^h u(x) = \frac{u(x + he_j) - u(x)}{h},$$

where e_j is the j-th element of the standard basis of $\mathbb{R}^n$.

Exercise 15.39. Let $\Omega \subset \mathbb{R}^n$ open and $h \neq 0$. Show that, if $u, v : \Omega \to \mathbb{C}$, then, for every $j \in \{1, \dots, n\}$ and $x \in \Omega_{|h|}$,

$$\Delta_j^h (uv)(x) = \Delta_j^h u(x) v(x + he_j) + u(x) \Delta_j^h v(x). \tag{153}$$

◊

Exercise 15.40. Let $\Omega \subset \mathbb{R}^n$ open and $h \neq 0$. Show that, if $u, v : \Omega \to \mathbb{C}$ and $\operatorname{spt}(v) \subset \Omega_{|h|}$, then, for every $j \in \{1, \dots, n\}$,

$$\int_\Omega \Delta_j^h u(x) v(x) \,\mathrm{d}x = -\int_\Omega u(x) \Delta_j^{-h} v(x) \,\mathrm{d}x. \tag{154}$$

Notice that, since $\operatorname{spt}(v) \subset \Omega_{|h|}$, we have: first, if $x \in \operatorname{spt}(v)$, then $x + he_j \in \Omega$; second, if $x \in \operatorname{spt}(\Delta_j^{-h} v)$, then $x \in \operatorname{spt}(v)$ or $x - he_j \in \operatorname{spt}(v)$, which implies $x \in \Omega$. ◊

Exercise 15.41. Let $\Omega \subset \mathbb{R}^n$ open and $j \in \{1, \dots, n\}$. Show that, if $\phi \in C_c^1(\Omega)$, then

$$\lim_{h \to 0} \|\Delta_j^h \phi - \partial_j \phi\|_{L^\infty(\Omega)} = 0. \tag{155}$$

◊

Exercise 15.42. Let $\Omega \subset \mathbb{R}^n$ open. Show that, if $u \in L^1_{\mathrm{loc}}(\Omega)$, then, for every $j \in \{1, \dots, n\}$ and every $\phi \in C_c^\infty(\Omega)$,

$$\lim_{j \to 0} \int_\Omega \Delta_j^h u(x) \phi(x) \,\mathrm{d}x = \partial_j u[\phi], \tag{156}$$

where we see $u \in \mathscr{D}'(\Omega)$ as a distribution with distributional derivative $\partial_j u \in \mathscr{D}'(\Omega)$. In other words, $\Delta_j^h u \to \partial_j u$ in $\mathscr{D}'(\Omega)$.

Solution. Fix u, j and ϕ. Since $\operatorname{spt}(\phi)$ is compact, there is $\epsilon > 0$ so that $\phi \in C_c^\infty(\Omega_\epsilon)$. Since $\operatorname{spt}(\phi) \Subset \Omega_\epsilon$, then there is $\delta > 0$ such that $B(\operatorname{spt}(\phi), \delta) = \bigcup_{x \in \operatorname{spt}(\phi)} B(x, \delta) \Subset \Omega_\epsilon$. Therefore,

$$\begin{aligned}
&\limsup_{h \to 0} \left| \int_\Omega \Delta_j^h u(x) \phi(x) \,\mathrm{d}x - \partial_j u[\phi] \right| \\
&\overset{(154)}{=} \limsup_{h \to 0} \left| -\int_\Omega u(x) \Delta_j^{-h} \phi(x) \,\mathrm{d}x + \int_\Omega u(x) \partial_j \phi(x) \,\mathrm{d}x \right| \\
&\leq \limsup_{h \to 0} \int_\Omega |u(x)| \cdot | - \Delta_j^{-h} \phi(x) + \partial_j \phi(x)| \,\mathrm{d}x \\
&\overset{(\text{Hölder})}{\leq} \limsup_{h \to 0} \|u\|_{L^1(B(\operatorname{spt}(\phi), \delta))} \cdot \| - \Delta_j^{-h} \phi(x) + \partial_j \phi(x)\|_{L^\infty(\Omega)} \\
&\overset{(155)}{=} 0.
\end{aligned}$$

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Proposition 15.43. *Let $\Omega \subset \mathbb{R}^n$ be an open set and $p \in (1, +\infty]$. For every $u \in L^p(\Omega)$ and $j \in \{1, \dots, n\}$, the following are equivalent:*

(i) $\partial_j u \in L^p(\Omega)$*;*

(ii) there exists $C \in \mathbb{R}$ such that, for every $\epsilon > 0$, $\limsup_{h \to 0} \|\Delta_j^h u\|_{L^p(\Omega_\epsilon)} \leq C$.

Moreover, for every $\epsilon > 0$ and h with $|h| < \epsilon$,

$$\|\Delta_j^h u\|_{L^p(\Omega_\epsilon)} \leq \|\partial_j u\|_{L^p(\Omega)}. \tag{157}$$

Proof. $\boxed{(i) \Rightarrow (ii)\text{: case } p \in (1,\infty).}$ Let $u \in C^1(\Omega)$ and $\epsilon > 0$. For $|h| < \epsilon$ and $p \in [1, +\infty]$, we have

$$\begin{aligned}
\int_{\Omega_\epsilon} |\Delta_j^h u(x)|^p \,\mathrm{d}x &= \int_{\Omega_\epsilon} \left|\frac{u(x+he_j)-u(x)}{h}\right|^p \mathrm{d}x \\
&\overset{(163)}{=} \int_{\Omega_\epsilon} \left|\int_0^1 \partial_j u(x+the_j)\,\mathrm{d}t\right|^p \mathrm{d}x \\
&\overset{(\text{Hölder})}{\leq} \int_{\Omega_\epsilon}\int_0^1 |\partial_j u(x+the_j)|^p \,\mathrm{d}t\,\mathrm{d}x \\
&= \int_0^1 \int_{\Omega_\epsilon} |\partial_j u(x+the_j)|^p \,\mathrm{d}x\,\mathrm{d}t \\
&\leq \int_0^1 \int_{\Omega_{\epsilon-|h|}} |\partial_j u(x)|^p \,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega_{\epsilon-|h|}} |\partial_j u(x)|^p \,\mathrm{d}x.
\end{aligned} \tag{158}$$

We have shown (157) for smooth functions. Next, let $u \in L^p(\Omega)$ with $\partial_j u \in L^p(\Omega)$. Let $\{\rho_\eta\}_{\eta>0}$ be a standard family of mollifiers and define $u_k = u * \rho_{1/k} : \Omega_{1/k} \to \mathbb{C}$. For $1/k < \epsilon - |h|$, we have $\Omega_\epsilon \subset \Omega_{\epsilon-|h|} \subset \Omega_{1/k}$ and thus, from (158),

$$\int_{\Omega_\epsilon} |\Delta_j^h u_k(x)|^p \,\mathrm{d}x \leq \int_{\Omega_{\epsilon-|h|}} |\partial_j u_k(x)|^p \,\mathrm{d}x. \tag{159}$$

Moreover, we know that $u_k \to u$ and $\partial_j u_k \to \partial_j u$ in $L^p(\Omega_{\epsilon-|h|})$. We thus have, for $|h| < \epsilon$,

$$\begin{aligned}
&\|\Delta_j^h u_k - \Delta_j^h u\|_{L^p(\Omega_\epsilon)} \\
&= \left(\int_{\Omega_\epsilon} \left|\frac{u_k(x+he_j)-u_k(x)}{h} - \frac{u(x+he_j)-u(x)}{h}\right|^p \mathrm{d}x\right)^{1/p} \\
&\overset{(\text{Minkowski})}{\leq} \left(\int_{\Omega_\epsilon} \left|\frac{u_k(x+he_j)-u(x+he_j)}{h}\right|^p \mathrm{d}x\right)^{1/p} + \left(\int_{\Omega_\epsilon} \left|\frac{u(x)-u_k(x)}{h}\right|^p \mathrm{d}x\right)^{1/p} \\
&\leq \frac{2}{h}\|u_k - u\|_{L^p(\Omega_{\epsilon-|h|})}.
\end{aligned}$$

Therefore, for each $h \neq 0$ with $|h| < \epsilon$ fixed, $\Delta_j^h u_k \to \Delta_j^h u$ in $L^p(\Omega_{\epsilon-|h|})$ as $k \to \infty$. This convergence allows us to extend the estimate (159) to the limit. We obtain (157) and thus (ii).

$\boxed{(i) \Rightarrow (ii)\text{: case } p = \infty.}$ Let $u \in C^1(\Omega)$ and $\epsilon > 0$. For $|h| < \epsilon$ and $x \in \Omega_\epsilon$, we have

$$\begin{aligned}
\sup_{x\in\Omega_\epsilon} |\Delta_j^h u(x)| &= \sup_{x\in\Omega_\epsilon} \left|\frac{u(x+he_j)-u(x)}{h}\right| \\
&= \sup_{x\in\Omega_\epsilon} \left|\int_0^1 \frac{\nabla u(x+the_j)\cdot he_j}{h}\,\mathrm{d}t\right| \\
&\leq \sup_{x\in\Omega} |\partial_j u(x)| = \|\partial_j u\|_{L^\infty(\Omega)}.
\end{aligned} \tag{160}$$

We have shown (157) for smooth functions. Next, let $u \in L^\infty(\Omega)$ with $\partial_j u \in L^\infty(\Omega)$. Like in the previous case, define $u_k = u * \rho_{1/k} : \Omega_{1/k} \to \mathbb{C}$, so that $u_k \in C^1(\Omega_{1/k})$. For $1/k < \epsilon - |h|$, we have $\Omega_\epsilon \subset \Omega_{\epsilon-|h|} \subset \Omega_{1/k}$. [...]

Using Exercise 15.45, we have that, for fixed h and j, there is a full measure set $E \subset \Omega_{\epsilon-|h|}$, such that, for every $x \in E$, we have $\lim_{k\to\infty} u_k(x) = u(x)$, and $\lim_{k\to\infty} u_k(x+he_j) = u(x+he_j)$. Moreover, $\|\partial_j u_k\|_{L^\infty(\Omega_{1/k})} \leq \|\partial_j u\|_{L^\infty(\Omega)}$. Therefore, for every $x \in E$,

$$\begin{aligned}
|\Delta_j^h u(x)| &= \left|\frac{u(x+he_j)-u(x)}{h}\right| \\
&= \lim_{k\to\infty} \left|\frac{u_k(x+he_j)-u_k(x)}{h}\right| \\
&\overset{(160)}{\leq} \limsup_{k\to\infty} \|\partial_j u_k\|_{L^\infty(\Omega_{1/k})} \\
&\leq \|\partial_j u\|_{L^\infty(\Omega)}.
\end{aligned}$$

We obtain (157) and thus (ii) for $p = \infty$ too.

$\boxed{(ii) \Rightarrow (i).}$ Fix $\epsilon > 0$. If $p \in (1, +\infty)$, then $L^p(\Omega_\epsilon)$ is the dual of $L^{p'}(\Omega_\epsilon)$, where $p' = \frac{p}{p-1}$. If $p = \infty$, then the same is true with $p' = 1$.

Since $\limsup_{h\to 0} \|\Delta_j^h u\|_{L^p(\Omega_\epsilon)} \le C < \infty$, we can apply the Banach–Alaoglu Theorem 15.46. Therefore, there is a sequence $\{h_k\}_{k\in\mathbb{N}} \subset (0, \epsilon)$ with $\lim_{k\to\infty} h_k = 0$ and there is $v_\epsilon \in L^p(\Omega_\epsilon)$ such that

$$\|v_\epsilon\|_{L^p(\Omega_\epsilon)} \le C \tag{161}$$

and, for every $\phi \in L^{p'}(\Omega)$,

$$\lim_{k\to\infty} \int_{\Omega_\epsilon} \Delta_j^{h_k} u(x)\phi(x)\,\mathrm{d}x = \int_{\Omega_\epsilon} v_\epsilon(x)\phi(x)\,\mathrm{d}x. \tag{162}$$

In particular, $C_c^\infty(\Omega_\epsilon) \subset L^{p'}(\Omega_\epsilon)$. Combining (156) with (162), we get $v_\epsilon = \partial_j u$, that is, the distributional derivative $\partial_j u$ of u on Ω_ϵ is in fact a function in $L^p(\Omega_\epsilon)$.

By the locality of distirbutions, we obtain that $\partial_j u$ is a function on Ω that satisfies, by (161), $\|\partial_j u\|_{L^p(\Omega_\epsilon)} \le C$ for all $\epsilon > 0$. Therefore, $\|\partial_j u\|_{L^p(\Omega)} \le C$. □

Recall that the dual of L^1 is L^∞, although the dual of L^∞ is not L^1

Exercise 15.44. Suppose $u \in C^1(\Omega)$, $x \in \Omega$, $v \in \mathbb{R}^n$ such that $x + tv \in \Omega$ for all $t \in [0, 1]$. Then

$$u(x + v) - u(x) = \int_0^1 \nabla u(x + tv) \cdot v\,\mathrm{d}t. \tag{163}$$

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Exercise 15.45. Let $\Omega \subset \mathbb{R}^n$, $\{\rho_\eta\}_{\eta>0}$ a standard family of mollifiers, and $u \in L^1_{\mathrm{loc}}(\Omega)$. Define $u_\epsilon = u * \rho_\epsilon : \Omega_\epsilon \to \mathbb{C}$. Show that, for every $x \in \Omega_\epsilon$,

$$u_\epsilon(x) \le \|u\|_{L^\infty(B(x,\epsilon))}.$$

Moreover, for almost every $x \in \Omega$, $\lim_{\epsilon\to 0} u_\epsilon(x) = u(x)$.

Find an example where $u \in L^\infty(\Omega)$ but, for every $h > 0$, $\liminf_{\epsilon\to 0} \|u_\epsilon - u\|_{L^\infty(\Omega_h)} > 0$.

Hint for the example. Take $\Omega = (-1, 1) \subset \mathbb{R}$ (or $\Omega = \mathbb{R}$), and $u = \mathbb{1}_{(0,1)}\cdots$ ◊

Theorem 15.46 (Banach–Alaoglu Theorem)**.** *Let $(V, \|\cdot\|)$ be a normed space, and let $(V', \|\cdot\|_*)$ be the dual space endowed with the operator norm*

$$\|\alpha\|_* = \sup\{\alpha[x] : x \in V,\ \|x\| \le 1\}, \qquad \forall \alpha \in V'.$$

If $\{\alpha_k\}_{k\in\mathbb{N}} \subset V'$ is a bounded sequence, that is, $\sup_{k\in\mathbb{N}} \|\alpha_k\|_* < \infty$, then there exists a unique $\alpha_\infty \in V'$ which is the weak limit of α_k, that is $\alpha_k \overset{*}{\rightharpoonup} \alpha_\infty$. More explicitly, for every $x \in V$,*

$$\lim_{k\to\infty} \alpha_k[x] = \alpha_\infty[x].$$

Moreover,

$$\|\alpha_\infty\|_* \le \liminf_{k\to\infty} \|\alpha_k\|_*.$$

Proof. See `https://en.wikipedia.org/wiki/Banach\T1\textendashAlaoglu_theorem` . □

Theorem 15.47 (Characterization of Sobolev spaces with differential quotients)**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $u \in L^1_{\mathrm{loc}}(\Omega)$. For every $p \in (1, +\infty]$, the following are equivalent*

(i) $u \in W^{1,p}(\Omega)$;
(ii) $u \in L^p(\Omega)$ *and there exists $C > 0$ such that, for every $j \in \{1, \dots, n\}$ and every $\epsilon > 0$, $\limsup_{h\to 0} \|\Delta_j^h u\|_{L^p(\Omega_\epsilon)} \le C$.*

Proof. This is a direct consequence of Proposition 15.43. □

§15.13. Differentiability a.e. for $n < p \le \infty$. For $\Omega \subset \mathbb{R}^n$ open, a function $u : \Omega \to \mathbb{C}$ is *differentiable* at $x \in \Omega$ if there exists a linear function $\alpha : \mathbb{R}^n \to \mathbb{C}$ such that

$$\lim_{y \to x} \frac{|u(y) - u(x) - \alpha[y - x]|}{|y - x|} = 0. \tag{164}$$

If u is differentiable at x, the linear function α in (164) is unique and is of the form $\alpha[y - x] = \nabla_{cl} u(x) \cdot (y - x)$ for a vector $\nabla_{cl} u(x) \in \mathbb{C}^n$. We call this vector $\nabla_{cl} u(x)$ the *classical gradient* of u at x. If u is C^1, then it is clear that $\nabla_{cl} u(x) = \nabla u(x)$ for every x, where $\nabla u(x)$ is the gradient we have used so far. When $u \in L^1_{\text{loc}}(\Omega)$, then we have a distributional gradient $\nabla u \in \mathscr{D}'(\Omega)^n$, which may be an element of $L^p(\Omega)$ in the case of Sobolev functions, but we don't know a priori that $\nabla u(x) = \nabla_{cl} u(x)$.

In fact, it can be that $u \in W^{1,p}(\Omega)$ is not differentiable anywhere. Indeed, recall from your course in Analysis, or simply prove it from (164), that if u is differentiable at x, then u is continuous at x. It follows that the function constructed in Exercise 15.7, which is nowhere continuous, is nowhere differentiable although it has a weak gradient in L^p.

Theorem 15.48. *Let $n \ge 2$ and $p \in (n, +\infty]$. If $\Omega \subset \mathbb{R}^n$ is open and $u \in W^{1,p}_{\text{loc}}(\Omega)$, then, for almost every $x \in \Omega$, u is differentiable at x and $\nabla u(x) = \nabla_{cl} u(x)$.*

Proof. First, assume $p \in (n, +\infty)$. By Theorem 15.25, we can assume u continuous. By Exercise 15.49, for almost every $x \in \Omega$, we have

$$\lim_{r \to 0} \fint_{B(x,r)} |\nabla u(y) - \nabla u(x)|^p \,\mathrm{d}x = 0. \tag{165}$$

Let $x \in \Omega$ be a point with (165). Let $R > 0$ be such that $B(x, 4R) \Subset \Omega$. Define $v : B(x, 4R) \to \mathbb{C}$ by

$$v(y) = u(y) - u(x) - \nabla u(x) \cdot (y - x), \qquad \forall y \in B(x, 4R).$$

Clearly, we have $v \in W^{1,p}(B(x, 4R)) \cap C^0(B(x, 4R))$ and $\nabla v(y) = \nabla u(y) - \nabla u(x)$. Therefore, applying Theorem 15.25 to v, we obtain for every $y \in B(x, R)$,

$$\begin{aligned}
\frac{|u(y) - u(x) - \nabla u(x) \cdot (y - x)|}{|x - y|} &= \frac{|v(y) - v(x)|}{|x - y|} \\
&\overset{(148)}{\le} C|x - y|^{-\frac{n}{p}} \|\nabla u(y) - \nabla u(x)\|_{L^p(B(o,4R))} \\
&= C \left(\frac{\omega_n^{1/n} 4R}{|x - y|} \right)^{n/p} \left(\fint_{B(x,4R)} |\nabla u - \nabla u(x)|^p \,\mathrm{d}x \right)^{\frac{1}{p}}.
\end{aligned}$$

If we take $R = |x - y|$, this estimate combined with (165), gives

$$\lim_{y \to x} \frac{|u(y) - u(x) - \nabla u(x) \cdot (y - x)|}{|x - y|} = 0,$$

that is, $\nabla u(x) = \nabla_{cl} u(x)$.

Finally, if $p = \infty$, then we clearly have $W^{1,\infty}_{\text{loc}}(\Omega) \subset W^{1,p}_{\text{loc}}(\Omega)$ for all $p \in (n, \infty)$. So, we apply the result we have just proven. □

Exercise 15.49 (A variant of Lebesgue differentiation theorem)**.** Let $\Omega \subset \mathbb{R}^n$ open. Show that, if $f \in L^p(\Omega)$, then for almost every $x \in \Omega$ we have

$$\lim_{r \to 0} \fint_{B(x,r)} |f(y) - f(x)|^p \,\mathrm{d}y = 0.$$

Solution. Look at [13, §1.5.7&§1.1.8]. It can be proven for $f \in L^p(\mu)$ with (X, d, μ) a doubling metric measure space. ♢

§15.14. $p = \infty$: Lipschitz functions. Let $\Omega \subset \mathbb{R}^n$ open. A function $u : \Omega \to \mathbb{C}$ is L-Lipschitz for some $L \in \mathbb{R}$ if

$$\forall x, y \in \Omega, \qquad |u(x) - u(y)| \le L|x - y|.$$

Lipschitz functions are clearly continuous

Theorem 15.50. *Let $\Omega \subset \mathbb{R}^n$ open and convex. For every $u \in L^\infty(\Omega)$ and $L \in \mathbb{R}$, the following are equivalent* *(i)* u *has a* L*-Lipschitz representative, in the sense that, for almost every* $x, y \in \Omega$, $|u(x) - u(y)| \le L|x - y|$;

(ii) $u \in W^{1,\infty}(\Omega)$ *and* $\|\nabla u\|_{L^\infty(\Omega)} \le L$.

As as consequence, if $u \in W^{1,\infty}(\Omega)$*, then* u *is* $\|\nabla u\|_{L^\infty(\Omega)}$*-Lipschitz.*

Proof. $\boxed{(i) \Rightarrow (ii).}$ If $u : \Omega \to \mathbb{C}$ is L-Lipschitz, then, for every $x \in \Omega$, $j \in \{1, \dots, n\}$ and $h \neq 0$ with $|h| < \mathrm{dist}(x, \partial\Omega)$,

$$|\Delta_j^h u(x)| = \left| \frac{u(x + he_j) - u(x)}{h} \right| \le L.$$

Therefore, by Theorem 15.47, we obtain $u \in W^{1,\infty}(\Omega)$. Moreover, using also Proposition 15.43, we have, for almost every $x \in \Omega$,

$$|\nabla u(x)| \le \max\{|\partial_j u(x)| : j \in \{1, \dots, n\}\} \le L.$$

So, $\|\nabla u\|_{L^\infty(\Omega)} \le L$.

$\boxed{(ii) \Rightarrow (i).}$ We can use Morrey's inequality, Theorem 15.25, to show that if $u \in W^{1,\infty}(\Omega)$ then u is Lipschitz. We just need to notice that $u \in W^{1,p}(B)$ for every $B \Subset \Omega$ and $p \in (n, \infty)$. Then, we take the limit $p \to \infty$ in (148), where the constant $C = C(p, n)$ remains bounded. However, in this way it is not evident that u is $\|\nabla u\|_{L^\infty(\Omega)}$-Lipschitz.

So, we can apply another argument, just by usual mollification. One easily sees that $u_\epsilon = u * \rho_\epsilon \in C^\infty(\Omega_\epsilon)$ is a smooth function with $|\nabla u_\epsilon(x)| \le \|\nabla u\|_{L^\infty(\Omega)}$ for every $x \in \Omega_\epsilon$. Since u_ϵ is smooth, we can estimate for every $x, y \in \Omega$, using convexity,

$$|u_\epsilon(x) - u_\epsilon(y)| = \left| \int_0^1 \nabla u_\epsilon(x + t(y - x)) \cdot (y - x)\, \mathrm{d}t \right| \le \|\nabla u\|_{L^\infty(\Omega)} |x - y|.$$

Since $u_\epsilon \to u$ uniformly on compact sets, then $\lim_{\epsilon \to 0} |u_\epsilon(x) - u_\epsilon(y)| = |u(x) - u(y)|$. □

Exercise 15.51. In Theorem 15.50 we used convexity of the set Ω. Give an example of open set Ω that is connected but not convex where Theorem 15.50 fails. What can we say in any case? ♢

Theorem 15.52 (Rademacher Theorem)**.** *If* $\Omega \subset \mathbb{R}^n$ *is an open set and* $u : \Omega \to \mathbb{C}$ *is Lipschitz, then* u *is differentiable almost every where in* Ω.

Proof. This is a consequence of Theorem 15.50 and Theorem 15.48. □

§15.15. **Compactness theorems: Ascoli–Arzelà.** One of our main tools to prove compactness is the following standard result

Theorem 15.53 (Ascoli–Arzelà)**.** *Let* K *be a compact metric space and* $\mathscr{F} = \{f_k\}_{k \in \mathbb{N}} \subset C(K)$ *be a sequence of continuous functions* $K \to \mathbb{C}$*. Suppose that*

(1) $\mathscr{F}$ *is (equi)bounded, that is, there is* C *such that* $|f(x)| \le C$ *for all* $f \in \mathscr{F}$ *and all* $x \in K$.

(2) $\mathscr{F}$ *is equicontinuos, that is, for every* $\epsilon > 0$ *there exists* $\delta > 0$ *such that, if* $x, y \in K$ *and* $d(x, y) < \delta$*, then* $|f(x) - f(y)| < \epsilon$ *for all* $f \in \mathscr{F}$.

Then there exists a subsequence $\{f_{k_j}\}_{j \in \mathbb{N}} \subset \mathscr{F}$ *that converges uniformly on* K.

§15.16. **Compactness theorems: Rellich–Kondrachov for** $W_0^{1,p}(\mathbb{R}^n)$**.** Recall that a linear operator $L : A \to B$ between Banach spaces is *compact* if it maps bounded subsets of A to pre-compact subsets of B. In other words, L is a compact operator if, for every bounded sequence $\{a_k\}_{k \in \mathbb{N}} \subset A$, there is a subsequence $\{La_{k_j}\}_{j \in \mathbb{N}}$ converging in B. We shall prove that for certain p and q, the "identity map" $W_0^{1,p}(\Omega) \to L^q(\Omega)$ is a compact linear operator.

Remark 15.54. The proof is taken from [5, §5.5.7]. For a similar proof, see [3, Theorem 4.26]. For more general statements, see [1, Theorem 6.3].

Lemma 15.55. *Let $n \geq 1$ and $\{\rho_\epsilon\}_{\epsilon>0}$ be a standard family of mollifiers. Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and define $u_\epsilon = u * \rho_\epsilon$. For every $\Omega \subset \mathbb{R}^n$, we have*

$$\int_\Omega |u_\epsilon(x) - u(x)| \,\mathrm{d}x \leq \epsilon \int_{B(\Omega,\epsilon)} |\nabla u|(x) \,\mathrm{d}x, \tag{166}$$

where, we recall, $B(\Omega, \epsilon) = \bigcup_{x\in\Omega} B(x, \epsilon)$.

Proof. First, assume $u \in C^1(\mathbb{R}^n)$. Notice that, for every $x \in \mathbb{R}^n$,

$$\begin{aligned}
|u_\epsilon(x) - u(x)| &= \left|\int_{\mathbb{R}^n} \rho_\epsilon(y)(u(x-y) - u(x)) \,\mathrm{d}y\right| \\
&= \left|\int_{\mathbb{R}^n} \rho_\epsilon(y) \int_0^1 \nabla u(x-ty)\cdot(-y) \,\mathrm{d}t \,\mathrm{d}y\right| \\
&\leq \int_{\mathbb{R}^n}\int_0^1 \rho_\epsilon(y)|\nabla u|(x-ty)|y| \,\mathrm{d}t \,\mathrm{d}y \\
[\mathrm{spt}(\rho_\epsilon) \subset B(0,\epsilon)] &\leq \epsilon \int_{\mathbb{R}^n}\int_0^1 \rho_\epsilon(y)|\nabla u|(x-ty) \,\mathrm{d}t \,\mathrm{d}y.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_\Omega |u_\epsilon(x) - u(x)| \,\mathrm{d}x &\leq \epsilon \int_\Omega \int_{\mathbb{R}^n}\int_0^1 \rho_\epsilon(y)|\nabla u|(x-ty) \,\mathrm{d}t \,\mathrm{d}y \,\mathrm{d}x \\
&\leq \epsilon \int_{\mathbb{R}^n}\int_0^1 \rho_\epsilon(y) \left(\int_\Omega |\nabla u|(x-ty) \,\mathrm{d}x\right) \mathrm{d}t \,\mathrm{d}y \\
&\leq \epsilon \int_{\mathbb{R}^n}\int_0^1 \rho_\epsilon(y) \,\mathrm{d}t \,\mathrm{d}y \left(\int_{B(\Omega,\epsilon)} |\nabla u|(x-ty) \,\mathrm{d}x\right) \\
&= \epsilon \int_{B(\Omega,\epsilon)} |\nabla u|(x-ty) \,\mathrm{d}x.
\end{aligned}$$

Second, consider $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ arbitrary. For every $\eta \in (0,1)$, we have (166) for the smooth function u_η. On the one hand, since $(u_\eta)_\epsilon = (u * \rho_\eta) * \rho_\epsilon = (u * \rho_\epsilon) * \rho_\eta$, we have

$$\lim_{\eta\to 0} \|(u_\eta)_\epsilon(x) - u_\eta\|_{L^1(\Omega)} = \lim_{\eta\to 0} \|(u_\epsilon(x) - u)_\eta\|_{L^1(\Omega)} \overset{(??)}{=} \|u_\epsilon(x) - u\|_{L^1(\Omega)}.$$

On the other hand,

$$\lim_{\eta\to 0} \|\nabla u_\eta\|_{L^1(B(\Omega,\epsilon))} \overset{(??)}{=} \|\nabla u\|_{L^1(B(\Omega,\epsilon))}.$$

Therefore, we obtain (166) for u too. □

Exercise 15.56. Extend (166) of Lemma 15.55 to all $u \in W^{1,1}(\mathbb{R}^n)$. What can we say for $u \in W^{1,1}_0(\Omega)$, when $\Omega \subset \mathbb{R}^n$ is open. ◊

Lemma 15.57 (Interpolation inequality for L^p norms)**.** *Let $1 \leq p \leq q \leq r \leq \infty$ and $\theta \in [0,1]$ such that*

$$\frac{1}{q} = \frac{\theta}{p} + \frac{1-\theta}{r}. \tag{167}$$

Then, whenever μ is a measure and u is μ-measurable function,

$$\|u\|_{L^q(\mu)} \leq \|u\|^\theta_{L^p(\mu)} \cdot \|u\|^{1-\theta}_{L^r(\mu)}. \tag{168}$$

In particular, if $u \in L^p(\mu) \cap L^r(\mu)$, then $u \in L^q(\mu)$ for all $q \in [p, r]$.

Proof. The identity (167) is equivalent to

$$1 = \frac{q\theta}{p} + \frac{q(1-\theta)}{r} = \frac{1}{p/(q\theta)} + \frac{1}{r/(q(1-\theta))},$$

that is, $\frac{p}{q\theta}$ and $\frac{r}{q(1-\theta)}$ are Hölder conjugate exponents. Since they are both belong to $[1, +\infty]$, we can apply the Hölder inequality:

$$\int |u|^q \,\mathrm{d}\mu = \int |u|^{\theta q}|u|^{(1-\theta)q} \,\mathrm{d}\mu$$

$$\overset{\text{(Hölder)}}{\leq} \left(\int |u|^{\theta q \frac{p}{q\theta}} \, \mathrm{d}\mu \right)^{\frac{q\theta}{p}} \left(\int |u|^{(1-\theta)q \frac{r}{q(1-\theta)}} \, \mathrm{d}\mu \right)^{\frac{q(1-\theta)}{r}}$$
$$= \left(\|u\|_{L^p(\mu)}^{\theta} \cdot \|u\|_{L^r(\mu)}^{1-\theta} \right)^q .$$

So, we get (168) by exponentiating by $\frac{1}{q}$. □

Lemma 15.58. *Let $n \geq 1$ and $\{\rho_\epsilon\}_{\epsilon>0}$ be a standard family of mollifiers. Let $u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and define $u_\epsilon = u * \rho_\epsilon$. Then, for every $p \in [1,n)$ and $q \in [1,p^*)$, there exists $C \in \mathbb{R}$ such that*

$$\|u_\epsilon - u\|_{L^q(\mathbb{R}^n)} \leq C\epsilon^\theta \|\nabla u\|_{L^1(\mathbb{R}^n)}^{\theta} \cdot \|\nabla u\|_{L^p(\mathbb{R}^n)}^{1-\theta}, \tag{169}$$

where $\theta \in [0,1]$ is such that $\frac{1}{q} = \theta + \frac{1-\theta}{p^*}$.

Proof. First, we apply the Interpolation inequality for L^p norms, Lemma 15.57, to get

$$\|u_\epsilon - u\|_{L^q(\mathbb{R}^n)} \leq \|u_\epsilon - u\|_{L^1(\mathbb{R}^n)}^{\theta} \cdot \|u_\epsilon - u\|_{L^{p^*}(\mathbb{R}^n)}^{1-\theta}.$$

Second, we apply the bound (166) from Lemma 15.55 to the first term $\|u_\epsilon - u\|_{L^1(\mathbb{R}^n)}$, and the Gagliardo–Nirenberg–Sobolev Inequality (147) from Theorem 15.22 or Corollary 15.24 to the second term $\|u_\epsilon - u\|_{L^{p^*}(\mathbb{R}^n)}$. We thus get

$$\|u_\epsilon - u\|_{L^1(\mathbb{R}^n)}^{\theta} \cdot \|u_\epsilon - u\|_{L^{p^*}(\mathbb{R}^n)}^{1-\theta} \leq \left(\epsilon \|\nabla u\|_{L^1(\mathbb{R}^n)}\right)^{\theta} \cdot \left(C_{GNS} \|\nabla u\|_{L^p(\mathbb{R}^n)}\right)^{1-\theta}.$$

We have thus obtained (169). □

Remark 15.59. In the inequalities (166) and (169), it might happen that the right-hand side is $+\infty$. The inequalities are still true, although they don't provide any extra information.

Proposition 15.60. *Let $n \geq 2$, $p \in [1,n)$ and $q \in [1,p^*)$. Suppose that $\{u_k\}_{k\in\mathbb{N}} \subset W^{1,p}(\mathbb{R}^n)$ is a sequence such that there is $R > 0$ with $\mathrm{spt}(u_k) \subset B(0,R)$ for all $k \in \mathbb{N}$. Suppose also that there exists $M \in \mathbb{R}$ such that $\|u_k\|_{W^{1,p}(\mathbb{R}^n)} \leq M$ for all $k \in \mathbb{N}$. Then, there exists a subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ that is converging in $L^q(\mathbb{R}^n)$.*

Proof. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a standard family of mollifiers. and define $u_k^\epsilon = u_k * \rho_\epsilon$.

We claim that there are $C \in \mathbb{R}$ and $\theta \in [0,1]$ such that, for every $\epsilon > 0$ and $k \in \mathbb{N}$,

$$\|u_k^\epsilon - u_k\|_{L^q(\mathbb{R}^n)} \leq C\epsilon^\theta. \tag{170}$$

Indeed, if we apply (169) from Lemma 15.58, we get

$$\begin{aligned}
\|u_k^\epsilon - u_k\|_{L^q(\mathbb{R}^n)} &\overset{(169)}{\leq} C_{(169)}\epsilon^\theta \|\nabla u\|_{L^1(\mathbb{R}^n)}^{\theta} \cdot \|\nabla u\|_{L^p(\mathbb{R}^n)}^{1-\theta} \\
&\overset{\text{(Hölder)}}{\leq} C_{(169)}\epsilon^\theta \|\nabla u\|_{L^p(\mathbb{R}^n)}^{\theta} \cdot \mathscr{L}^n(B(0,R))^{\theta/p'} \cdot \|\nabla u\|_{L^p(\mathbb{R}^n)}^{1-\theta} \\
&= C_{(169)}\epsilon^\theta (\omega_n R^n)^{\theta/p'} \cdot \|\nabla u\|_{L^p(\mathbb{R}^n)} \\
&\leq C_{(169)} (\omega_n R^n)^{\theta/p'} M \epsilon^\theta .
\end{aligned}$$

So, we have proven (170).

Notice that, for every $\epsilon > 0$, $k \in \mathbb{N}$, and $x \in \mathbb{R}^n$,

$$\begin{aligned}
|u_k^\epsilon(x)| &\leq \int_{\mathbb{R}^n} \rho_\epsilon(y) |u(x-y)| \, \mathrm{d}y \\
&\overset{\text{(Hölder)}}{\leq} \|\rho_\epsilon\|_{L^{p'}(\mathbb{R}^n)} \|u\|_{L^p(\mathbb{R}^n)} \\
&= \frac{\|\rho_1\|_{L^p(\mathbb{R}^n)}}{\epsilon^{n/p}} \|u\|_{L^p(\mathbb{R}^n)} \\
&\leq \frac{\|\rho_1\|_{L^p(\mathbb{R}^n)}}{\epsilon^{n/p}} M,
\end{aligned}$$

and, similarly,

$$\begin{aligned}
|\nabla u_k^\epsilon(x)| &\leq \int_{\mathbb{R}^n} \rho_\epsilon(y)|\nabla u(x-y)|\,\mathrm{d}y \\
&\overset{(\text{Hölder})}{\leq} \|\rho_\epsilon\|_{L^{p'}(\mathbb{R}^n)}\|\nabla u\|_{L^p(\mathbb{R}^n)} \\
&\overset{(174)}{=} \frac{1}{\epsilon^{n\frac{p'-1}{p'}}}\|\rho_1\|_{L^p(\mathbb{R}^n)}\|\nabla u\|_{L^p(\mathbb{R}^n)} \\
&= \frac{\|\rho_1\|_{L^p(\mathbb{R}^n)}}{\epsilon^{n/p}}\|\nabla u\|_{L^p(\mathbb{R}^n)} \\
&\leq \frac{\|\rho_1\|_{L^p(\mathbb{R}^n)}}{\epsilon^{n/p}} M.
\end{aligned}$$

It follows that, for each $\epsilon \in (0,1)$ fixed, the family $\{u_k^\epsilon\}_{k\in\mathbb{N}} \subset C_c^0(B(0,R+1))$ is bounded and equicontinuous. By Ascoli–Arzelà Theorem 15.53, $\{u_k^\epsilon\}_{k\in\mathbb{N}}$ is pre-compact in $C_c^0(B(0,R+1))$.

We apply a diagonal argument. First, let $\{u^1_{k^1_j}\}_{j\in\mathbb{N}}$ be a subsequence of $\{u^1_k\}_{k\in\mathbb{N}}$ that is converging uniformly on $\bar{B}(0,R+1)$ to a function $v_1 \in C_c^0(\bar{B}(0,R+1))$. Next, for every $m \in \mathbb{N}_{\geq 2}$, there is a subsequence $\{u^{1/m}_{k^m_j}\}_{j\in\mathbb{N}}$ of $\{u^{1/m}_{k^{m-1}_j}\}_{j\in\mathbb{N}}$ that is converging uniformly on $\bar{B}(0,R+1)$. to some $v_m \in C_c^0(\bar{B}(0,R+1))$. Notice that, for every $m \in \mathbb{N}_{\geq 1}$,

$$\limsup_{j\to\infty}\|u^{1/m}_{k^{m-1}_j} - v_m\|_{L^p(\mathbb{R}^n)} \leq \limsup_{j\to\infty}\|u^{1/m}_{k^{m-1}_j} - v_m\|_{L^\infty(\bar{B}(0,R+1))}\mathscr{L}^n(\Omega)^{1/p} = 0.$$

Therefore, $\{u^{1/m}_{k^m_j}\}_{j\in\mathbb{N}}$ is converging to v_m also in $L^q(\mathbb{R}^n)$.

We claim that $\{u_{k^m_m}\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\mathbb{R}^n)$. Indeed, let $\delta > 0$. Then there is $L \in \mathbb{N}$ such that

$$C(1/L)^\theta < \delta. \tag{171}$$

Next, since $\{u^{1/L}_{k^L_j}\}_{j\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\mathbb{R}^n)$, then there is $N \in \mathbb{N}$ such that, for every naturals $a,b > N$,

$$\forall a,b > N, \qquad \|u^{1/L}_{k^L_a} - u^{1/L}_{k^L_b}\|_{L^q(\mathbb{R}^n)} < \delta. \tag{172}$$

We also assume $N > L$. Since we have taken always subsequences, if $m > L$ then $\{k^m_a\}_{a>N} \subset \{k^L_a\}_{a>N}$. Therefore, (172) implies

$$\forall a,b > N, \qquad \|u^{1/L}_{k^a_a} - u^{1/L}_{k^b_b}\|_{L^q(\mathbb{R}^n)} < \delta. \tag{173}$$

It follows that, for every $a,b > N$,

$$\begin{aligned}
\|u_{k^a_a} - u_{k^b_b}\|_{L^q(\mathbb{R}^n)} &\leq \|u_{k^a_a} - u^{1/L}_{k^a_a}\|_{L^q(\mathbb{R}^n)} + \|u^{1/L}_{k^a_a} - u^{1/L}_{k^b_b}\|_{L^q(\mathbb{R}^n)} + \|u^{1/L}_{k^b_b} - u_{k^b_b}\|_{L^q(\mathbb{R}^n)} \\
&\overset{(170)\&(173)}{\leq} 2C(1/L)^\theta + \delta \\
&\overset{(171)}{\leq} 3\delta.
\end{aligned}$$

We have proven our claim, that $\{u_{k^m_m}\}_{m\in\mathbb{N}}$ is a Cauchy sequence in $L^q(\mathbb{R}^n)$. □

Exercise 15.61. Let $\{\rho_\epsilon\}_{\epsilon>0}$ be a standard family of mollifiers on $\mathbb{R}^n$. Show that, for every $p \in [1,+\infty]$,

$$\|\rho_\epsilon\|_{L^p(\mathbb{R}^n)} = \frac{1}{\epsilon^{n\frac{p-1}{p}}}\|\rho_1\|_{L^p(\mathbb{R}^n)}. \tag{174}$$

Solution.

$$\begin{aligned}
\int_{\mathbb{R}^n} \rho_\epsilon(y)^p\,\mathrm{d}y &= \int_{\mathbb{R}^n}\left(\frac{\rho_1(y/\epsilon)}{\epsilon^n}\right)^p\mathrm{d}y \\
[z = y/\epsilon,\ \ \mathrm{d}z = \mathrm{d}y/\epsilon^n] &= \frac{1}{\epsilon^{n(p-1)}}\int_{\mathbb{R}^n}\rho_1(z)^p\,\mathrm{d}z.
\end{aligned}$$

◊

Exercise 15.62. In Proposition 15.60, can we take the subsequence $\{u_{k_j}\}_{j\in\mathbb{N}}$ independent of q? ♢

Exercise 15.63. Proposition 15.60 uses a general fact in metric spaces. Let (X,d) be a complete metric space and $\{x_k^m\}_{k\in\mathbb{N},m\in\mathbb{N}\cup\{\infty\}}\subset X$. Suppose that:

(1) for every $k\in\mathbb{N}$, $\{x_k^m\}_{m\in\mathbb{N}}$ converges to x_k^∞, uniformly in k; explicitly, for every $\epsilon>0$ there exists $N\in\mathbb{N}$ such that, for every $m>N$ and for every $k\in\mathbb{N}$, $d(x_k^m,x_k^\infty)<\epsilon$;
(2) For every $m\in\mathbb{N}$ (but not for $m=\infty$), the set $\{x_k^m\}_{k\in\mathbb{N}}$ is pre-compact in X.

Show that $\{x_k^\infty\}_{k\in\mathbb{N}}$ is pre-compact in X. ♢

§15.17. Extension domains. An open set $\Omega\subset\mathbb{R}^n$ is called an *extension domain* if, for every $p\in[1,+\infty]$, there exists $T_p:W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^n)$ such that

(1) $T_pu|_\Omega=u$, for all $u\in W^{1,p}(\Omega)$;
(2) T_p is bounded, that is, there is C_p such that $\|T_pu\|_{W^{1,1}(\mathbb{R}^n)}\leq C_p\|u\|_{W^{1,p}(\Omega)}$, for all $u\in W^{1,p}(\Omega)$.

An example of a set that is NOT and extension domain, is the so called *slit disk*:

$$\Omega=\{(x,y)\in\mathbb{R}^2:x^2+y^2<1\}\setminus\{(x,0):x\geq 0\}.$$

However, there holds the following result

Theorem 15.64. *If $\Omega\subset\mathbb{R}^n$ is an open set with C^1 boundary, then it is an extension domain.*

In fact, Theorem 15.64 can be pushed to Lipschitz boundary.

Proposition 15.65. *Suppose $\Omega\subset\mathbb{R}^n$ is an extension domain and $\Omega'\subset\mathbb{R}^n$ is an open set such that $B(\Omega,r)\subset\Omega'$, for some $r>0$. Then, for every $p\in[1,+\infty]$ there exists a continuous extension operator $T_p:W^{1,p}(\Omega)\to W_0^{1,p}(\Omega')$.*

Proof. Let $\zeta\in C_c^\infty(\Omega')$ such that $0\leq\zeta\leq 1$ and

$$\bar\Omega\subset\operatorname{int}\{\zeta=1\}\subset\operatorname{spt}(\zeta)\subset\Omega'.$$

Since $B(\Omega,r)\subset\Omega'$, we can take ζ with $\|\nabla\zeta\|_{L^\infty(\mathbb{R}^n)}\leq 2/r$. Let $\tilde T_p::W^{1,p}(\Omega)\to W^{1,p}(\mathbb{R}^n)$ be a bounded extension operator given by Ω being and extension domain.

We claim that $T_p:u\mapsto\zeta\tilde T_pu$ is the wanted operator. To prove our claim, we need to show that $u\mapsto\zeta u$ defines a continuous operator $W^{1,p}(\mathbb{R}^n)\to W_0^{1,p}(\Omega')$. Clearly we have

$$\|\zeta u\|_{L^p(\Omega')}\leq\|u\|_{L^p(\mathbb{R}^n)}.$$

Moreover,

$$\begin{aligned}\|\nabla(\zeta u)\|_{L^p(\Omega')}&\leq\|\zeta\nabla u\|_{L^p(\mathbb{R}^n)}+\|u\nabla\zeta\|_{L^p(\mathbb{R}^n)}\\&\leq\|\nabla u\|_{L^p(\mathbb{R}^n)}+\frac{2}{r}\|u\|_{L^p(\mathbb{R}^n)}\\&\leq(1+2/r)\|u\|_{W^{1,p}(\mathbb{R}^n)}.\end{aligned}$$

The claim is proven. □

Proposition 15.66. *Let $\Omega\subset\Omega'\subset\mathbb{R}^n$ and $p\in[1,+\infty)$. Define $T_p:W_0^{1,p}(\Omega)\to L^p(\mathbb{R}^n)$ as $T_pu=\mathbb{1}_\Omega u$, i.e., T_p extends functions to zero outside Ω. Then T_p is a continuous operator $W_0^{1,p}(\Omega)\to W^{1,p}(\Omega')$. In fact,*

$$\forall u\in W^{1,p}(\Omega)\qquad\|\mathbb{1}_\Omega u\|_{W^{1,p}(\Omega')}=\|u\|_{W^{1,p}(\Omega)}\tag{175}$$

Proof. If $u\in C_c^1(\Omega)$, then $\|T_pu\|_{W^{1,p}(\Omega')}=\|u\|_{W^{1,p}(\Omega)}$. Since $C_c^1(\Omega)$ is dense in $W_0^{1,p}(\Omega)$, then we have (175). □

Remark 15.67. By Propositions 15.65 and 15.66, if Ω is a bounded extension domain, we can assume that the extension operator takes values in the space of functions with compact support in some fixed neighborhood of Ω.

§15.18. Extension of Sobolev inequalities to Extension domains. Many properties of $W^{1,p}(\mathbb{R}^n)$ extend to extension domains.

§15.19. Poincaré inequality for $W_0^{1,p}(\Omega)$.

Theorem 15.68. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ a bounded open set. For every $p \in [1, +\infty)$ there is $C \in \mathbb{R}$ such that,*

$$\forall u \in W_0^{1,p}(\Omega), \qquad \|u\|_{L^p(\Omega)} \leq C\|\nabla u\|_{L^p(\Omega)}. \tag{176}$$

Remark 15.69. The inequality in (176) can also be written as

$$\forall u \in W_0^{1,p}(\Omega), \qquad \int_\Omega |u|^p \, \mathrm{d}x \leq C^p \int_\Omega |\nabla u|^p \, \mathrm{d}x.$$

Lemma 15.70. *Let $1 \leq a < b < \infty$. If $\Omega \subset \mathbb{R}^n$ is a measurable set, then*

$$\forall u \in L^1_{\mathrm{loc}}(\Omega), \qquad \|u\|_{L^a(\Omega)} \leq |\Omega|^{\frac{b-a}{ba}} \|u\|_{L^b(\Omega)}, \tag{177}$$

where $|\Omega| = \mathscr{L}^n(\Omega)$ is the volume of Ω.

Proof. Since $b/a > 1$, we can apply the Hölder inequality:

$$\int_\Omega |u|^a \, \mathrm{d}x \overset{(\text{Hölder})}{\leq} \left(\int_\Omega |u|^{a\frac{b}{a}} \, \mathrm{d}x \right)^{\frac{a}{b}} \cdot \left(\int_\Omega 1^{\frac{\frac{b}{a}}{\frac{b}{a}-1}} \right)^{\frac{\frac{b}{a}-1}{\frac{b}{a}}} = \left(\int_\Omega |u|^b \, \mathrm{d}x \right)^{\frac{a}{b}} \cdot |\Omega|^{\frac{b-a}{b}}.$$

□

Proof of Theorem 15.68. We shall prove (176) assuming $u \in C^1_c(\Omega)$, to obtain the general statement by approximation in $W_0^{1,p}(\Omega)$.

Fix $p \in [1, +\infty)$. Recall that, if $q \in [1, n)$, then $q^* = \frac{nq}{n-q}$. Since $\lim_{q \to n^-} q^* = +\infty$, there is some $q \in [1, n)$ such that $q^* > p$. Let $u \in C^1_c(\Omega)$ Therefore, using Theorem 15.22,

$$\begin{aligned} \|u\|_{L^p(\Omega)} &\overset{(177)}{\leq} |\Omega|^{\frac{q^*-p}{q^*p}} \|u\|_{L^{q^*}(\Omega)} \\ &\overset{(147)}{\leq} |\Omega|^{\frac{q^*-p}{q^*p}} \|\nabla u\|_{L^q(\Omega)}. \end{aligned}$$

If $p < n$, then we can take $q = p$ already (because $p^* > p$), and (176) is proven with $C = |\Omega|^{\frac{p^*-p}{p^*p}} = |\Omega|^{\frac{1}{n}}$. If $p \geq n$, then, for each appropriate $q \in (1, n)$ we have $q < p$ and thus

$$\|\nabla u\|_{L^q(\Omega)} \overset{(177)}{\leq} |\Omega|^{\frac{p-q}{pq}} \|\nabla u\|_{L^q(\Omega)}.$$

This implies (176) with $C = |\Omega|^{\frac{q^*-p}{q^*p} + \frac{p-q}{pq}}$. □

§15.20. Extra that might be added in the future.

- Poincaré inequality for extension domains

Part 4. Application of Sobolev spaces theory to PDE

16. Elliptic PDEs

§16.1. Setting. Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open. We are interested in functions $u : \Omega \to \mathbb{C}$ such that

$$Lu(x) = \mathrm{div}(A(x)\nabla u(x)) + b(x) \cdot \nabla u(x) + c(x)u(x) \overset{!}{=} f(x), \tag{178}$$

where $A(x) \in \mathbb{C}^{n\times n}$ is a $n \times n$-complex matrix, $b(x) \in \mathbb{C}^n$ and $c(x), f(x) \in \mathbb{C}$, for each $x \in \Omega$.

How do we interpret the formula (178)? We may use distributional calculus: in such a case, we need to make sense of the products $A\nabla u = \sum_{jk} A_{jk}\partial_j u$, $b \cdot \nabla u = \sum_j b_j \partial_j u$, and cu. If the coefficients A, b and c are smooth, then we can consider $Lu = f$ for $u, f \in \mathscr{D}'(\Omega)$; see **§13.15**. Another possibility is that we consider $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$: in such a case, all derivatives of u are L^1_{loc} functions and thus, if the coefficients A, b and c are bounded in L^∞, then the products are still of class L^1_{loc} and thus $\mathrm{div}(A\nabla u) = \sum_{jk} \partial_j(A_{jk}\partial_k u) \in \mathscr{D}'$.

Definition 16.1. The *standard conditions on L* are:

(1) $A \in L^\infty(\Omega; \mathbb{C}^{n\times n})$, such that $A(x)$ is symmetric for every $x \in \Omega$ and there are $0 < \lambda \leq \Lambda < \infty$ with

$$\forall x \in \Omega,\ \forall \xi \in \mathbb{R}^n \qquad \lambda|\xi|^2 \leq \langle \xi, A(x)\xi\rangle = \sum_{i,j=1}^n A_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2. \tag{179}$$

(2) $b \in L^\infty(\Omega; \mathbb{C}^n)$, $c \in L^\infty(\Omega; \mathbb{C})$.

Condition (179) is called *ellipticity.* So, L as in (178) is *elliptic* if A satisfy (179).

Notice that, under these conditions, the formula for $Lu(x)$ written in (178) is not well founded. Indeed, even taking distributional derivatives, if A is only in L^∞, then the product $A(x) \cdot \nabla u(x)$ is not a well defined distribution. However, we have written L in *divergence form* so that we can say the following: If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, then we define

$$\begin{aligned} &Lu = f \text{ in } \Omega,\\ &\Leftrightarrow\\ &\forall \phi \in C^\infty_c(\Omega) \quad \int_\Omega \Big(\langle A\nabla u, \nabla\phi\rangle + b\cdot\nabla u\phi + cu\phi\Big)\,\mathrm{d}x = \int_\Omega f\phi\,\mathrm{d}x,\\ &\Leftrightarrow\\ &\forall \phi \in C^\infty_c(\Omega) \quad \int_\Omega \Big(\sum_{jk} A_{jk}\,\partial_j u\,\partial_k\phi + (\sum_j b_j\,\partial_j u + c\,u)\,\phi\Big)\,\mathrm{d}x = \int_\Omega f\phi\,\mathrm{d}x. \end{aligned} \tag{180}$$

Exercise 16.2. Let $p \in [1, \infty)$ and set $p' = \frac{p}{p-1}$ the Hölder conjugate of p. Show that, if $u \in W^{1,p}(\Omega)$ satisfies $Lu = f$ as in (180) with $f \in L^p(\Omega)$, then

$$\forall \phi \in W^{1,p'}_{\mathrm{loc}}(\Omega) \quad \int_\Omega \Big(\langle A\nabla u, \nabla\phi\rangle + b\cdot\nabla u\phi + cu\phi\Big)\,\mathrm{d}x = \int_\Omega f\phi\,\mathrm{d}x.$$

♢

Exercise 16.3 (??)**.** If $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and A, b, c are bounded and A elliptic, and $Lu = f \in L^1_{\mathrm{loc}}(\Omega)$, then $\mathrm{div}(A\nabla u) \in L^1_{\mathrm{loc}}(\Omega)$. Show that $u \in W^{2,1}_{\mathrm{loc}}(\Omega)$. ♢

§16.2. The Sobolev space H^m. Let $n \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^n$ open. If $m \in \mathbb{N}$, the Sobolev space $H^m(\Omega)$ is nothing else than the Sobolev space with integral exponent 2, that is,

$$H^m(\Omega) = W^{m,2}(\Omega), \quad H^m_{\mathrm{loc}}(\Omega) = W^{m,2}_{\mathrm{loc}}(\Omega), \quad H^m_0(\Omega) = W^{m,2}_0(\Omega).$$

These spaces are in fact Hilbert spaces, when endowed with a correct norm. We will focus on $m = 1$ (and $m = 0$, which is L^2).

For $u, v \in H^1(\Omega)$, we define

$$\begin{aligned} \langle u, v\rangle_{H^1(\Omega)} &:= \langle\!\langle u, v\rangle\!\rangle\\ &:= \int_\Omega (\langle\nabla u, \nabla\bar v\rangle + u\cdot\bar v)\,\mathrm{d}x\\ &= \langle\nabla u, \nabla v\rangle_{L^2(\Omega)} + \langle u, v\rangle_{L^2(\Omega)}. \end{aligned} \tag{181}$$

Proposition 16.4. *Let $n \geq 1$ and $\Omega \subset \mathbb{R}^n$ open. Then the bilinear map $\langle\langle \cdot, \cdot \rangle\rangle$ defined in* (181) *makes $H^1(\Omega)$ into a Hilbert space, with norm*

$$\|u\|_{H^1(\Omega)} = \sqrt{\langle u, u \rangle_{H^1(\Omega)}}, \qquad \forall u \in H^1(\Omega),$$

which is biLipschitz equivalent to the Sobolev norm $\|\cdot\|_{W^{1,2}(\Omega)}$.

Proof. For every $x \in \Omega$, and every $u, v \in H^1(\Omega)$, we have $|\langle \nabla u(x), \nabla \bar{v}(x) \rangle| \leq |\nabla u(x)| \cdot |\nabla \bar{v}(x)|$. Notice that, if $u \in H^1(\Omega)$, then

$$\begin{aligned} \|u\|^2_{H^1(\Omega)} &= \langle u, u \rangle_{H^1(\Omega)} \\ &= \langle \nabla u, \nabla u \rangle_{L^2(\Omega)} + \langle u, u \rangle_{L^2(\Omega)} \\ &= \|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_L(\Omega)^2, \end{aligned}$$

that is,

$$\|u\|_{H^1(\Omega)} = \sqrt{\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^2_L(\Omega)^2}$$

Since there are constants $c, C \in (0, +\infty)$ such that, for every $a, b \in \mathbb{R}^2$, $c\sqrt{a^2 + b^2} \leq |a| + |b| \leq C\sqrt{a^2 + b^2}$, then

$$c\|u\|_{H^1(\Omega)} \leq \|u\|_{W^{1,2}(\Omega)} = \|\nabla u\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)} \leq C\|u\|_{H^1(\Omega)}.$$

See also Exercise 15.1. □

Remark 16.5. Proposition 16.4 has already a quite interesting consequence for PDE. Let $f \in L^2(\Omega)$ and define $T_f : H^1(\Omega) \to \mathbb{C}$, $T_f u = \langle f, u \rangle_{L^2(\Omega)}$. The operator T_f is in fact a bounded operator $L^2(\Omega) \to \mathbb{C}$, therefore it is bounded also on $H^1(\Omega)$; Explicitly, we have $|T_f u| \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}$. Since $H^1(\Omega)$ is a Hilbert space, the Riesz Representation Theorem implies that there exists a unique $u \in H^1(\Omega)$ such that, for all $\phi \in H^1(\Omega)$, $T_f \phi = \langle u, \phi \rangle_{H^1(\Omega)}$. In particular,

$$\forall \phi \in C^\infty_c(\Omega), \qquad \int_\Omega f\bar{\phi}\,\mathrm{d}x = \int_\Omega ((\nabla u \cdot \nabla \bar{\phi}) + u\bar{\phi})\,\mathrm{d}x. \tag{182}$$

The property (182) has a distributional interpretation: since $\int_\Omega (\nabla u \cdot \nabla \bar{\phi})\,dx = \sum_{j=1}^n \partial_j u[\partial_j \bar{\phi}] = -\triangle u[\phi]$, then (182) is equivalent in $\mathscr{D}'(\Omega)$ to

$$-\triangle u + u = f. \tag{183}$$

We have thus proven that, for every $f \in L^2(\Omega)$, there exists a unique $u \in H^1(\Omega)$ that is a distributional solution to (183).

Exercise 16.6. Show that, if $\Omega \subset \mathbb{R}^n$ is an open and bounded set (or with finite volume), then, for every $\lambda \in (-\infty, 0]$, there exists a unique solution to

$$\begin{cases} -\triangle u = \lambda u, \\ u \in H^1_0(\Omega). \end{cases}$$

Since such solution must be $u = 0$, you have shown that the half-line $(-\infty, 0]$ is not in the spectrum of $-\triangle$. ♢

§16.3. An alternative scalar product on $H^1_0(\Omega)$. If Ω is a bounded open subset of $\mathbb{R}^n$,[8] the Poincaré inequality from Theorem 15.68, implies that

$$\langle\langle u, v \rangle\rangle := \langle \nabla u, \nabla v \rangle_{L^2(\Omega)}, \qquad \forall u, v \in H^1_0(\Omega), \tag{184}$$

is a Hilbert scalar product on $H^1_0(\Omega)$. Indeed,

$$\frac{1}{2C}\|u\|_{L^2(\Omega)} + \frac{1}{2}\|\nabla u\|_{L^2(\Omega)} \overset{(176)}{\leq} \|\nabla u\|_{L^2(\Omega)} = \sqrt{\langle\langle u, u \rangle\rangle} \leq \|u\|_{H^1(\Omega)},$$

which shows that the quasi-norm $u \mapsto \|\nabla u\|_{L^2(\Omega)}$ is a norm bi-Lipschitz equivalent to $u \mapsto \|u\|_{H^1(\Omega)}$.

As we did in Remark 16.5, we can deduce an existence and uniqueness result for a PDE. Indeed, if $f \in L^2(\Omega)$, then the operator $T_f \phi := \langle f, \phi \rangle_{L^2(\Omega)}$ is bounded on $H^1_0(\Omega)$.

[8] Connected? only finite measure? **TODO**

Therefore, by the Riesz Representation Theorem, there exists a unique $u \in H^1_0(\Omega)$ such that

$$\forall \phi \in C^\infty_c(\Omega), \qquad \langle\!\langle u, \phi \rangle\!\rangle = T_f \phi.$$

Distributionally, this reads as

$$\triangle u = f \text{ in } \Omega.$$

We have proven the following theorem:

Theorem 16.7. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. For every $f \in L^2(\Omega)$, the boundary problem*

$$\begin{cases} \triangle u = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \text{ i.e., } u \in H^1_0(\Omega), \end{cases}$$

has a unique solution $u \in H^1_0(\Omega)$.

§16.4. The dual of H^1. The dual space of H^m is, by definition, the space

$$H^{-m}(\Omega) := (H^m(\Omega))', \qquad \text{and} \qquad H_0^{-m}(\Omega) := (H^m_0(\Omega))'.$$

Let us focus on $m = 1$. Being a Hilbert space, the dual of H^m is canonically isomorphic to H^m itself. However, it is useful to keep the two spaces distinct. The main reason, in my view, is that we want to see $L^2(\Omega)$ as a subspace of the dual of $H^1(\Omega)$. Indeed, if $f \in L^2(\Omega)$, then, as we have seen above, $u \mapsto \int_\Omega uf \,\mathrm{d}x$ is an element of $H^m(\Omega)'$ (or $H^m_0(\Omega)'$). However, it is clear that $L^2(\Omega) \not\subset H^1(\Omega)$: in fact, $H^1(\Omega) \hookrightarrow L^2(\Omega)$.

For more discussions, see
`https://math.stackexchange.com/questions/314113/dual-space-of-h1`.

§16.5. Functional Analysis: Lax–Milgram Theorem. See also [15, Aufgabe V.6.18].

If X is a Banach space, we denote by X' its topological dual Banach space, and, for $\xi \in X'$ and $x \in X$, we write the pairing $\xi[x]$ as ${}_{X'}\langle \xi | x \rangle_X$.

The following theorem will replace the role of Riesz Theorem in the previous discussion. It is here written for Banach spaces: notice that there are Banach spaces that are isomorphic to their duals without being "hilbertable"; see `https://math.stackexchange.com/questions/65609/isometric-to-dual-implies-hilbertable`.

Theorem 16.8 (Lax–Milgram Theorem)**.** *Let $(X, \|\cdot\|_X)$ be a Banach space and $B : X \times X \to \mathbb{C}$ a bilinear map.*

(1) If B is bounded (i.e., continuous), that is, there is $\beta \in \mathbb{R}$ such that,

$$\forall u, v \in X, \qquad |B[u, v]| \le \beta \|u\|_X \cdot \|v\|_X, \tag{185}$$

then there is a bounded linear operator $T : X \to X'$ with $\|T\| \le \beta$ and such that

$$\forall u, v \in X, \qquad B[u, v] = {}_{X'}\langle Tu | \bar{v} \rangle_X. \tag{186}$$

(2) If B is bounded and coercive, that is, there exists $\delta > 0$

$$\forall u \in H, \qquad \mathfrak{Re}(B[u, u]) \ge \delta \|u\|^2, \tag{187}$$

then the linear operator $T : X \to X'$ is invertible and $\|T^{-1}\| \le \frac{1}{\delta}$.

Proof. Suppose B is bounded. If $u \in X$, then $v \mapsto B[u, v]$ is a continuous (thanks to (185)) linear functional $X \to \mathbb{C}$ and thus there exists a unique $Tu := w \in X'$ such that $B[u, v] = {}_{X'}\langle Tu | \bar{v} \rangle_X$ for all $v \in X$. It is easy to see that the so defined function $T : X \to X'$ is linear. We will next prove several properties of this operator T.

We claim that T is bounded, and

$$\|T\|_{X \to X'} \le \beta.$$

Indeed, if $u \in X$, then

$$\begin{aligned} \|Tu\|_{X'} &= \sup\{|{}_{X'}\langle Tu|v\rangle_X| : v \in X,\ \|v\|_X \le 1\} \\ &= \sup\{|B[u, \bar{v}]| : v \in X,\ \|v\|_X \le 1\} \\ &\overset{(185)}{\le} \sup\{\beta\|u\|_X \cdot \|v\|_X : v \in X,\ \|v\|_X \le 1\} \end{aligned}$$

$$= \beta\|u\|_X.$$

Therefore, we have proven the claim.

We claim that T is coercive, i.e.,

$$\forall u \in X, \qquad \|Tu\|_{X'} \geq \delta\|x\|_X. \tag{188}$$

Indeed, if $u \in X$, then

$$\delta\|x\|^2 \overset{(187)}{\leq} \mathfrak{Re}(B[x,\bar{x}]) \leq |B[x,\bar{x}]| = |_{X'}\langle Tu|v\rangle_X| \leq \|Tx\|_{X'}\|x\|_X.$$

Hence, (188) follows.

We claim that $\operatorname{im}(T) = T[X]$, the image of T, is closed in X'. Indeed, if $\alpha \in X'$ and if $\{u_k\}_{k\in\mathbb{N}} \subset X$ is a sequence such that $\lim_{k\to\infty} Tu_k = \alpha \in X'$, then, using coercivity of T, we have $\|u_j - u_k\|_X \overset{(188)}{\leq} \frac{1}{\delta}\|T[u_j - u_k]\|_{X'}$, and thus $\{u_k\}_{k\in\mathbb{N}}$ is a Cauchy sequence in X If $\lim_{k\to\infty} u_k = u$, then, by the boundedness of T, we have $\alpha = Tu \in \operatorname{im}(T)$.

We claim that $\operatorname{im}(T) = X'$. Since $\operatorname{im}(T)$ is closed, we only need to show that, if $v \in X$ annihilates to $\operatorname{im}(T)$, then $v = 0$ (thanks to Hahn–Banach Theorem). If $v \in H$ annihilates to $\operatorname{im}(T)$, then $0 = {}_{X'}\langle Tu|v\rangle_X$ for every $u \in X$. In particular, combining this with coercivity, we obtain $0 = {}_{X'}\langle Tv|v\rangle_X \overset{(188)}{\geq} \delta\|v\|_X^2$. Therefore, $v = 0$.

Finally, since T is a continuous, injective and surjective linear operator, its inverse is also continuous. □

§16.6. First Existence and Uniqueness result. We denote by $\mathbb{R}^{n\times n}$ the space of all $n \times n$ matrices. As such, if $A \in \mathbb{R}^{n\times n}$, then $|A|_\infty := \sup\{|Ax| : x \in \mathbb{R}^n,\ |x| \leq 1\}$. So, if $A \in L^\infty(\Omega;\mathbb{R}^{n\times n})$, then

$$\|A\|_{L^\infty(\Omega)} = \sup\{|A(x)|_\infty : x \in \Omega\}.$$

The scalar product $\langle\cdot,\cdot\rangle$ is in fact sequilinear and defined as

$$\forall x, y \in \mathbb{C}^n, \qquad \langle x, y\rangle = x \cdot \bar{y} = \sum_{j=1}^n x_j \bar{y}_j.$$

Theorem 16.9. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open and bounded. Let L be a second order linear differential operator in divergence form, that is,*

$$Lu = -\operatorname{div}(A\nabla u) + b \cdot \nabla u + cu, \tag{189}$$

where

$$A \in L^\infty(\Omega;\mathbb{R}^{n\times n}),\ b \in L^\infty(\Omega;\mathbb{C}^n),\ \text{and } c \in L^\infty(\Omega;\mathbb{C}). \tag{190}$$

Then L is a bounded linear operator $L : H_0^1(\Omega) \to H_0^{-1}(\Omega)$.

Suppose that there are $0 < \theta \leq \Theta < \infty$ such that

$$\forall x \in \Omega,\ \forall \xi \in \mathbb{C}^n, \qquad \theta|\xi|^2 \leq \langle A(x)\xi, \xi\rangle \leq \Theta|\xi|^2. \tag{191}$$

Then there is $\gamma \geq 0$ (in fact, the one for which holds (196)) such that for all $\lambda \in \mathbb{C}$ with $\mathfrak{Re}(\lambda) \geq \gamma$, the operator $L+\lambda : H_0^1(\Omega) \to H_0^{-1}(\Omega)$ is bounded and invertible. In particular, for every $f \in H_0^{-1}(\Omega)$ there exists a unique weak solution to the boundary problem

$$\begin{cases} Lu + \lambda u = f & \text{in } \Omega, \\ u \in H_0^1(\Omega). \end{cases} \tag{192}$$

Moreover, denoting by $\iota : H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ the standard embedding, the operator $K_\lambda = \iota \circ (L - \lambda)^{-1}|_{L^2(\Omega)} : L^2(\Omega) \to L^2(\Omega)$ is bounded, linear, and compact.

For proving Theorem 16.9, we will study the bilinear form $\mathscr{E}_L : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{C}$,

$$\mathscr{E}_L[u,v] = \int_\Omega (\langle A\nabla u, \nabla v\rangle + (\nabla u \cdot b + cu)\bar{v})\ \mathrm{d}x. \tag{193}$$

This bilinear form $\mathscr{E}_L$ is called the *Dirichlet form of L*.

Lemma 16.10 (Energy Estimates 1). *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open. Let L be a second order linear differential operator in divergence form as in* (189). *Assume that L has bounded coefficients, that is,* (190). *Define $\mathscr{E}_L : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ as in* (193).

Then, there is $\alpha \geq 0$ such that, for all $u, v \in H^1_0(\Omega)$,

$$\forall u, v \in H^1_0(\Omega), \qquad |\mathscr{E}_L[u, v]| \leq \alpha \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \tag{194}$$

In particular, the linear operator $L : H^1_0(\Omega) \to \mathscr{D}'$ is a continuous linear operator $L : H^1_0(\Omega) \to H^{-1}_0(\Omega)$ with

$$\forall u, v \in H^1_0(\Omega), \qquad \mathscr{E}_L[u, v] = {}_{H^{-1}_0(\Omega)}\langle Lu|\bar{v}\rangle_{H^1_0(\Omega)}. \tag{195}$$

Proof.

$$\begin{aligned} |\mathscr{E}_L[u, v]| &\leq \int_\Omega |\langle A\nabla u, \nabla v\rangle + \langle \nabla u, b\rangle \bar{v} + cu\bar{v}| \, \mathrm{d}x \\ &\leq \int_\Omega |A\nabla u||\nabla v| + |b||\nabla u||v| + |cu\bar{v}| \, \mathrm{d}x \\ &\leq \|A\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|b\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\qquad + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \\ &\leq (\|A\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)}) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

So, we have (194) with $\alpha = (\|A\|_{L^\infty(\Omega)} + \|b\|_{L^\infty(\Omega)} + \|c\|_{L^\infty(\Omega)})$.

By the Lax–Milgram Theorem 16.8, there is a continuous linear operator $T : H^1_0(\Omega) \to H^{-1}_0(\Omega)$ that satisfies the role of L in (195). We claim that $T = L$. Indeed, if $\phi \in \mathscr{D}(\Omega)$, then, for every $u \in H^1_0(\Omega)$ we have

$$\begin{aligned} Lu[\phi] &= {}_{\mathscr{D}'(\Omega)}\langle Lu|\phi\rangle_{\mathscr{D}(\Omega)} \\ &= (-\operatorname{div}(A\nabla u) + b \cdot \nabla u + cu)[\phi] \\ &= -\sum_{j=1}^n \partial_j (A\nabla u)_j[\phi] + \int_\Omega b \cdot \nabla u\phi + cu\phi \, \mathrm{d}x \\ &= \sum_{j=1}^n (A\nabla u)_j[\partial_j \phi] + \int_\Omega b \cdot \nabla u\phi + cu\phi \, \mathrm{d}x \\ &= \int_\Omega (\sum_{j=1}^n (A\nabla u)_j[\partial_j \phi] + b \cdot \nabla u\phi + cu\phi) \, \mathrm{d}x \\ &= \int_\Omega ((A\nabla u) \cdot \nabla\phi + b \cdot \nabla u\phi + cu\phi) \, \mathrm{d}x \\ &= \langle A\nabla u, \nabla\bar{\phi}\rangle_{L^2(\Omega)} + \langle (b \cdot \nabla u + cu), \bar{\phi})\rangle_{L^2(\Omega)} \\ &= \mathscr{E}_L[u, \bar{\phi}] \\ &= {}_{H^1_0(\Omega)'}\langle Tu, \phi\rangle_{H^1_0(\Omega)}. \end{aligned}$$

Since $\mathscr{D}(\Omega)$ is dense in $H^1_0(\Omega)$, we obtain that $Lu = Tu$ not just on $\mathscr{D}(\Omega)$, but also on $H^1_0(\Omega)$. □

Lemma 16.11 (Energy Estimates 2). *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open. Let L be a second order linear differential operator in divergence form as in* (189). *Assume that L has bounded coefficients, that is,* (190), *and that L is elliptic, that is,* (191). *Define $\mathscr{E}_L : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ as in* (193).

Then, there are $\beta > 0$ and $\gamma \geq 0$ such that, for all $u, v \in H^1_0(\Omega)$,

$$\beta \|\nabla u\|^2_{L^2(\Omega)} \leq \mathfrak{Re}(\mathscr{E}_L[u, u]) + \gamma \|u\|^2_{L^2(\Omega)}. \tag{196}$$

Proof. We have

$$\theta \int_\Omega |\nabla u|^2 \, \mathrm{d}x \overset{(191)}{\leq} \int_\Omega \langle A\nabla u, \nabla u\rangle \, \mathrm{d}x$$

$$
\begin{aligned}
&= \left| \mathscr{E}_L[u,u] - \int_\Omega \left(\langle \nabla u, b\rangle \bar{u} + c|u|^2 \right) \mathrm{d}x \right| \\
&\le |\mathscr{E}_L[u,u]| + \|b\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)} \\
&\overset{(197)}{\le} |\mathscr{E}_L[u,u]| + \|b\|_{L^\infty(\Omega)} \left(\epsilon \frac{\|\nabla u\|^2_{L^2(\Omega)}}{2} + \frac{1}{\epsilon} \frac{\|u\|^2_{L^2(\Omega)}}{2} \right) + \|c\|_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)} \\
&[\text{with } \epsilon = \frac{\theta}{\|b\|_{L^\infty(\Omega)}}] \\
&= |\mathscr{E}_L[u,u]| + \frac{\theta}{2} \|\nabla u\|^2_{L^2(\Omega)} + \frac{\|b\|^2_{L^\infty(\Omega)}}{2\theta} \|u\|^2_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|^2_{L^2(\Omega)} \\
&= |\mathscr{E}_L[u,u]| + \frac{\theta}{2} \|\nabla u\|^2_{L^2(\Omega)} + \left(\frac{\|b\|^2_{L^\infty(\Omega)}}{2\theta} + \|c\|_{L^\infty(\Omega)} \right) \|u\|^2_{L^2(\Omega)}.
\end{aligned}
$$

So, we have (196) with $\beta = \frac{\theta}{2}$ and $\gamma = \frac{\|b\|^2_{L^\infty(\Omega)}}{2\theta} + \|c\|_{L^\infty(\Omega)}$. □

Remark 16.12. Notice that, if c is real-valued and $c \ge c_0$ for some $c_0 \in \mathbb{R}$, then one can take $\gamma = \frac{\|b\|^2_{L^\infty(\Omega)}}{2\theta} + c_0$ in (196).

Lemma 16.13 (Cauchy inequality with ϵ)**.** *For every $a, b \in \mathbb{R}$, and for every $\epsilon > 0$,*

$$
ab \le \epsilon \frac{a^2}{2} + \frac{1}{\epsilon} \frac{b^2}{2}. \tag{197}
$$

Proof.

$$
\begin{aligned}
0 &\le \left(\sqrt{\epsilon} a - \frac{1}{\sqrt{\epsilon}} b \right)^2 \\
&= \epsilon a^2 + \frac{1}{\epsilon} b^2 - 2ab.
\end{aligned}
$$

□

Proof of Theorem 16.9. The conditions (190) on the coefficients of L easily imply that L is a continuous linear operator $H^1_0(\Omega) \to \mathscr{D}'(\Omega)$. Define $\mathscr{E}_L : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ as in (193). By Lemma 16.10, using again (190), the bilinear map $\mathscr{E}_L$ is bounded. By Theorem 16.8, there is a continuous operator $T : H^1_0(\Omega) \to H^{-1}_0(\Omega)$ such that (186) holds. However, see that (186) is equivalent to say that $T = L$. Indeed, if $u \in H^1_0(\Omega)$ and $\phi \in \mathscr{D}(\Omega) \subset H^1_0(\Omega)$,

$$
Lu[\phi] = \mathscr{E}_L[u, \phi] = {}_{H^{-1}_0(\Omega)}\langle Tu | \phi \rangle_{H^1_0(\Omega)},
$$

hence $Lu = Tu$ as elements of $\mathscr{D}'(\Omega)$. Therefore, L is a bounded linear operator $L : H^1_0(\Omega) \to H^{-1}_0(\Omega)$.

Next, we also assume the ellipticity condition (191). Let α, β and γ as in Lemmata 16.10 and 16.11, and let $\lambda \in \mathbb{C}$ with

$$
\mathfrak{Re}(\lambda) \ge \gamma. \tag{198}
$$

Define $B_\lambda : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ by

$$
\forall u, v \in H^1_0(\Omega), \qquad B_\lambda[u,v] = \mathscr{E}_L[u,v] + \lambda \langle u, v \rangle_{L^2(\Omega)}.
$$

On $H^1_0(\Omega)$, we consider the Hilbert scalar product $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ defined in (184), with norm $\| \cdot \|_{H^1_0(\Omega)} = \sqrt{\langle\!\langle \cdot, \cdot \rangle\!\rangle}$.

From Lemmata 16.10 and 16.11, we obtain that B_λ is a bounded and coercive bilinear map. Indeed, on the one hand, for every $u, v \in H^1_0(\Omega)$,

$$
|B_\lambda[u,v]| \overset{(194)}{\le} \alpha \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \le \alpha C \|u\|_{H^1_0(\Omega)} \|v\|_{H^1_0(\Omega)},
$$

where C is a constant of equivalence between the two norms on $H^1_0(\Omega)$. On the other hand, for every $u \in H^1_0(\Omega)$,

$$
\begin{aligned}
\mathfrak{Re}(B_\lambda[u,u]) &= \mathfrak{Re}(\mathscr{E}_L[u,u]) + \mathfrak{Re}(\lambda) \|u\|^2_{L^2(\Omega)} \\
&\overset{(196)}{\ge} \beta \|\nabla u\|^2_{L^2(\Omega)} + (\mathfrak{Re}(\lambda) - \gamma) \|u\|^2_{L^2(\Omega)}
\end{aligned}
$$

$$\overset{(198)}{\geq} \beta\|\nabla u\|^2_{L^2(\Omega)} = \beta\|u\|_{H^1_0(\Omega)}.$$

We apply Lax–Milgram Theorem 16.8: there exists a continuous, invertible linear operator $T_\lambda : H^1_0(\Omega) \to H^{-1}_0(\Omega)$ such that

$$\forall u, v \in H^1_0(\Omega) \qquad B_\lambda[u, v] = {}_{H^{-1}_0(\Omega)}\langle T_\lambda u|v\rangle_{H^1_0(\Omega)}.$$

We claim that $T_\lambda = T+\lambda$. Here we mean that $T_\lambda u = Tu+\lambda\iota(u)$, where $\iota : H^1_0(\Omega) \hookrightarrow L^2(\Omega)$ is the standard embedding. Indeed, for all $u, v \in H^1_0(\Omega)$,

$$\begin{aligned} {}_{H^{-1}_0(\Omega)}\langle Tu + \lambda u|v\rangle_{H^1_0(\Omega)} &= {}_{H^{-1}_0(\Omega)}\langle Tu|v\rangle_{H^1_0(\Omega)} + \lambda_{H^{-1}_0(\Omega)}\langle \iota(u)|v\rangle_{H^1_0(\Omega)} \\ &= \mathscr{E}_L[u, v] + \lambda\langle u, v\rangle_{L^2(\Omega)} = B_\lambda[u, v] \\ &= {}_{H^{-1}_0(\Omega)}\langle T_\lambda u|v\rangle_{H^1_0(\Omega)}. \end{aligned}$$

It follows that, for every $f \in H^{-1}_0(\Omega)$, the preimage $u = T^{-1}_\lambda[f]$ is the unique solution to (192).

Finally, the operator K is the composition of a continuous linear operator $(T - \lambda)^{-1}$ with a continuous linear compact operator $\iota : H^1_0(\Omega) \to L^2(\Omega)$. The compactness of ι comes from Rellich–Kondrachov Theorem **??**. □

§16.7. If you want kaos. Here I write something that may confuse the reader quite a lot. Read it at your own risk.

The pairing $\langle u, v\rangle_{L^2(\Omega)} = \int_\Omega u\bar{v}\,\mathrm{d}x$ is continuous on $H^1(\Omega)$, but it is NOT the Hilbert scalar product of H^1. For instance, the closure of $H^1(\Omega)$ with respect to the norm $\|\cdot\|_{L^2(\Omega)} = \sqrt{\langle u, v\rangle_{L^2(\Omega)}}$, is $L^2(\Omega)$. Continuity means that, if $f \in L^2(\Omega)$, then $u \mapsto \langle u, f\rangle_{L^2(\Omega)}$ is an element of the dual of $H^1_0(\Omega)$.

Riesz Theorem implies that there is some $v_f \in H^1(\Omega)$ such that

$$\forall u \in H^1(\Omega), \qquad \langle u, f\rangle_{L^2(\Omega)} \overset{!}{=} \langle \nabla u, \nabla v_f\rangle_{L^2(\Omega)} + \langle u, v_f\rangle_{L^2(\Omega)} \overset{\text{def}}{=} \langle u, v_f\rangle_{H^1(\Omega)}.$$

We have seen that, if Ω is bounded, then $\langle u, v\rangle_{H^1_0(\Omega)} = \int_\Omega\langle \nabla u, \nabla v\rangle\,\mathrm{d}x$ is a Hilbert scalar product on $H^1_0(\Omega)$. Again, Riesz Theorem implies that there exists $w_f \in H^1_0(\Omega)$ such that

$$\forall u \in H^1_0(\Omega), \qquad \langle u, f\rangle_{L^2(\Omega)} \overset{!}{=} \langle \nabla u, \nabla w_f\rangle_{L^2(\Omega)} \overset{\text{def}}{=} \langle u, w_f\rangle_{H^1_0(\Omega)}.$$

§16.8. Functional Analysis: Fredholm Alternative Theorem. Recall that, if X and Y are Banach spaces (e.g., Hilbert spaces), a linear operator $K : X \to Y$ is *compact operator* if K maps bounded sets to compact sets.

See also `https://terrytao.wordpress.com/2011/04/10/a-proof-of-the-fredholm-alternative/`

Theorem 16.14 (Fredholm Alternative)**.** *Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. Let $K : H \to H$ be a compact linear operator. For every $\lambda \in \mathbb{C}$,*

(1) $\ker(K - \lambda)$ *is finite dimensional;*
(2) $\mathrm{im}(K - \lambda)$ *is closed;*
(3) $\mathrm{im}(K - \lambda) = \ker(K^* - \bar{\lambda})^\perp$*;*
(4) $\dim(\ker(K - \lambda)) = \dim(\ker(K^* - \bar{\lambda}))$*;*
(5) $\ker(K - \lambda) = \{0\}$ *if and only if* $\mathrm{im}(K - \lambda) = H$.

§16.9. Second Existence and Uniqueness result.

Theorem 16.15. *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open and bounded. Let L be an operator as in Theorem 16.9 and define $B : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbb{C}$ as in (**??**). Define $\mathcal{N}, \mathcal{M} \subset H^1_0(\Omega)$ as*

$$\mathcal{N} = \{u \in H^1_0(\Omega) : \mathscr{E}_L[u, v] = 0\ \forall v \in H^1_0(\Omega)\},$$
$$\mathcal{M} = \{v \in H^1_0(\Omega) : \mathscr{E}_L[u, v] = 0\ \forall u \in H^1_0(\Omega)\}.$$

Then:

(1) Both $\mathcal{N}$ and $\mathcal{M}$ have finite dimension and $\dim(\mathcal{N}) = \dim(\mathcal{M})$.

(2) $\mathcal{N}$ *is the linear space of solutions to*

$$\begin{cases} Lu = 0, \\ u \in H^1_0(\Omega). \end{cases} \tag{199}$$

(3) for every $f \in L^2(\Omega)$*, the boundary value PDE*

$$\begin{cases} Lu = f, \\ u \in H^1_0(\Omega) \end{cases} \tag{200}$$

has a solution if and only if

$$\langle f, v\rangle_{L^2(\Omega)} = 0 \qquad \forall v \in \mathcal{M}.$$

(4) The affine space of solutions to (200)*, if not empty, has the same dimension of* $\mathcal{N}$*. In fact, given a solution* u_1 *to* (200)*, then* $u_1 + \mathcal{N}$ *is the space of solutions to* (200).

Proof. Let γ as in Theorem 16.9, i.e., as in Lemma 16.11. Define $K = T_\gamma^{-1} = (L+\gamma)^{-1}$, which is a compact bounded linear operator $L^2(\Omega) \to L^2(\Omega)$, as shown in Theorem 16.9.

Notice that, for every $u \in H^1_0(\Omega)$ and $f \in L^2(\Omega)$

$$\begin{aligned} u \text{ solves (200)} &\Leftrightarrow \mathscr{E}_L[u,v] = \langle f, v\rangle_{L^2(\Omega)} \qquad \forall v \in H^1_0(\Omega) \\ &\Leftrightarrow B_\gamma[u,v] = \langle f + \gamma u, v\rangle_{L^2(\Omega)} \qquad \forall v \in H^1_0(\Omega) \\ &\Leftrightarrow T_\gamma u = f + \gamma u \\ &\Leftrightarrow u = T_\gamma^{-1}(f+\gamma u) = Kf + \gamma K u \\ &\Leftrightarrow u - \gamma K u = Kf. \end{aligned} \tag{201}$$

So, considering the case $f = 0$, we obtain that u solves (199), if and only if $u \in \ker(\mathrm{Id} - \gamma K)$, that is,

$$\mathcal{N} = \ker(\mathrm{Id} - \gamma K).$$

Since γK is a compact operator $L^2(\Omega) \to L^2(\Omega)$, the Fredholm Alternative Theorem 16.14 implies that $\mathcal{N} = \ker(\mathrm{Id} - \gamma K)$ is finite dimensional.

Moreover, for each $f \in L^2(\Omega)$,

$$\begin{aligned} \text{(200) has a solution} &\overset{(201)}{\Leftrightarrow} Kf \in \mathrm{im}(\mathrm{Id} - \gamma K) \\ &\overset{\text{Thm 16.14}}{\Leftrightarrow} Kf \perp \ker(\mathrm{Id} - \gamma K^*) \\ &\overset{(*)}{\Leftrightarrow} f \perp \ker(\mathrm{Id} - \gamma K^*), \end{aligned}$$

where the equivalence $(*)$ is justified as follows: if $v \in \ker(\mathrm{Id} - \gamma K^*)$, i.e., $\gamma K^* v = v$, then $\gamma\langle Kf, v\rangle_{L^2(\Omega)} = \gamma\langle f, K^* v\rangle_{L^2(\Omega)} = \langle f, v\rangle_{L^2(\Omega)}$, so, $(*)$ holds, even when $\gamma = 0$.

We claim that

$$\mathcal{M} = \ker(\mathrm{Id} - \gamma K^*). \tag{202}$$

Notice that, similarly to what we have done in (201),

$$\begin{aligned} v \in \mathcal{M} &\Leftrightarrow \mathscr{E}_L[u,v] = 0 \quad \forall u \in H^1_0(\Omega) \\ &\Leftrightarrow B_\gamma[u,v] \overset{\text{def}}{=} B[u,v] + \gamma\langle u, \bar v\rangle_{L^2(\Omega)} = \langle u, \gamma \bar v\rangle_{L^2(\Omega)} \qquad \forall u \in H^1_0(\Omega) \\ &\Leftrightarrow {}_{H_0^{-1}(\Omega)}\langle T_\gamma u | \bar v\rangle_{H^1_0(\Omega)} = \langle u, \gamma \bar v\rangle_{L^2(\Omega)} \qquad \forall u \in H^1_0(\Omega) \\ &\Leftrightarrow {}_{H^1_0(\Omega)}\langle u | T_\gamma^\top \bar v\rangle_{H_0^{-1}(\Omega)} = \langle u, \gamma \bar v\rangle_{L^2(\Omega)} \qquad \forall u \in H^1_0(\Omega) \\ &\Leftrightarrow T_\gamma^\top \bar v = \gamma \bar v \\ &\Leftrightarrow \bar v - \gamma (T_\gamma^\top)^{-1} \bar v = 0 \\ &\Leftrightarrow v - \gamma \overline{(T_\gamma^\top)^{-1} \bar v} = 0 \end{aligned} \tag{203}$$

Here we have used the notation $(\cdot)^\top$ to denote the dual map: if $T : X \to Y$ is an operator between Banach spaces, then $T^\top : Y' \to X'$ is the dual map defined by

$$ {}_{Y'}\langle \zeta, Tx\rangle_Y = {}_{X'}\langle T^\top \zeta | x\rangle_X, \qquad \forall x \in X,\ \zeta \in Y'. \tag{204}$$

To complete the proof of our claim (202), we need to show that

$$\forall w \in L^2(\Omega), \qquad \overline{(T_\gamma^\top)^{-1}\bar{w}} = K^* w. \tag{205}$$

First, notice that $(T_\gamma^\top)^{-1} = (T_\gamma^{-1})^\top$, by an easy argument using directly (204):

$$\forall x \in X,\ \forall \xi \in X', \quad \langle \xi | x \rangle = \langle \xi | T^{-1}Tx \rangle = \langle T^\top (T^{-1})^\top \xi | x \rangle \quad \Rightarrow \quad T^\top (T^{-1})^\top = \mathrm{Id}_{X'}.$$

So, (205) reduces to showing $\overline{K^\top \bar{w}} = K^* w$ for all $w \in L^2(\Omega)$. More precisely, if $\tilde{K} = T_\gamma^{-1} : H_0^{-1}(\Omega) \to H_0^1(\Omega)$ (which is the operator from which K descends), then $\tilde{K}^\top : H_0^{-1}(\Omega) \to H_0^1(\Omega)$ and

$$\forall a, b \in H_0^{-1}(\Omega), \qquad {}_{H_0^1(\Omega)}\langle \tilde{K}a | b \rangle_{H_0^{-1}(\Omega)} = {}_{H_0^{-1}(\Omega)}\langle a | \tilde{K}^\top b \rangle_{H_0^1(\Omega)} \tag{206}$$

If $a, b \in L^2(\Omega) \subset H_0^{-1}(\Omega)$, then (206) says

$$\begin{aligned}
\langle a, K^* b \rangle_{L^2(\Omega)} &= \langle Ka, b \rangle_{L^2(\Omega)} \\
&= \langle \tilde{K}a, b \rangle_{L^2(\Omega)} \\
&= \int_\Omega \tilde{K}a(x)\bar{b}(x)\,\mathrm{d}x \\
&= {}_{H_0^1(\Omega)}\langle \tilde{K}a | \bar{b} \rangle_{H_0^{-1}(\Omega)} \\
&= {}_{H_0^{-1}(\Omega)}\langle a | \tilde{K}^\top \bar{b} \rangle_{H_0^1(\Omega)} \\
&= \int_\Omega a(x)\tilde{K}^\top \bar{b}(x)\,\mathrm{d}x \\
&= \langle a, \overline{\tilde{K}^\top \bar{b}} \rangle_{L^2(\Omega)}.
\end{aligned}$$

We can now complete the sequence of equivalences (203) with

$$\begin{aligned}
v \in \mathcal{M} &\overset{(203)}{\Leftrightarrow} v - \gamma\overline{(T_\gamma^\top)^{-1}\bar{v}} = 0 \\
&\Leftrightarrow v - \gamma K^* v = 0.
\end{aligned}$$

We have thus proven our claim (202).

With Theorem 16.14 and claim (202), we have proven all the statements of Theorem 16.15. □

Exercise 16.16. If $u \in \mathscr{N}$ means $Lu = 0$, what does $v \in \mathscr{M}$ mean? ◊

§16.10. Regularity. We assume that L has bounded coefficients, i.e., (191), and that L is elliptic, i.e., (190).

Lemma 16.17 (Caccioppoli Inequality[9])**.** *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open. Then there exists $C \in \mathbb{R}$ such that the following holds.*

If $u \in H^1(\Omega)$ is such that $Lu = f$ in Ω for some $f \in L^2(\Omega)$, i.e.,

$$\forall v \in H_0^1(\Omega), \qquad \mathscr{E}_L(u, v) = \langle u, f \rangle_{L^2(\Omega)},$$

then

$$\|\nabla u\|_{L^2(\Omega)}^2 \leq C(\|u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2). \tag{207}$$

Proof. Let $\Omega' \Subset \Omega$ and $\zeta \in C^\infty(\Omega; [0,1])$ be such that

$$\Omega' \subset \mathrm{int}\{\zeta = 1\} \subset \mathrm{spt}(\zeta) \Subset \Omega.$$

$$\begin{aligned}
\theta \|\nabla u\|_{L^2(\Omega')}^2 &= \theta \langle \nabla u, \nabla u \rangle_{L^2(\Omega')} \\
&\overset{(190)}{\leq} \langle A\nabla u, \nabla u \rangle_{L^2(\Omega')} \\
&\leq \langle A\nabla u, \nabla(\zeta u) \rangle_{L^2(\Omega)} \\
&\leq \mathscr{E}_L(u, \zeta u) - \langle b \cdot \nabla u + cu, \zeta u \rangle_{L^2(\Omega)} \\
&= \langle f - b \cdot \nabla u - cu, \zeta u \rangle_{L^2(\Omega)}
\end{aligned}$$

[9] https://en.wikipedia.org/wiki/Renato_Caccioppoli

$$
\begin{aligned}
&\overset{\text{(Hölder)}}{\le} (\|f\|_{L^2(\Omega)} + \|b\|_{L^\infty(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)})\|\zeta u\|_{L^2(\Omega)} \\
&\overset{(197)}{\le} \frac{\|f\|^2_{L^2(\Omega)}}{2} + \frac{\|\zeta u\|^2_{L^2(\Omega)}}{2} \\
&\quad + \frac{\theta}{2}\|\nabla u\|^2_{L^2(\Omega)} + \frac{\|b\|_{L^\infty(\Omega)}}{2\theta}\|\zeta u\|^2_{L^2(\Omega)} \\
&\quad + \|c\|_{L^\infty(\Omega)}\|u\|_{L^2(\Omega)}\|\zeta u\|_{L^2(\Omega)} \\
[\text{by } 0 \le \zeta \le 1] &\le \frac{\|f\|^2_{L^2(\Omega)}}{2} + \left(\frac{1}{2} + \frac{\|b\|_{L^\infty(\Omega)}}{2\theta} + \|c\|_{L^\infty(\Omega)}\right)\|u\|^2_{L^2(\Omega)} + \frac{\theta}{2}\|\nabla u\|^2_{L^2(\Omega)}.
\end{aligned}
$$

Therefore

$$
(208)\quad 2\|\nabla u\|^2_{L^2(\Omega')} - \|\nabla u\|^2_{L^2(\Omega)} \le \frac{2}{\theta}\frac{\|f\|^2_{L^2(\Omega)}}{2} + \big(1 + \|b\|_{L^\infty(\Omega)} + 2\theta\|c\|_{L^\infty(\Omega)}\big)\|u\|^2_{L^2(\Omega)}.
$$

Since the right-hand side of estimate (208) does not depend on Ω', if we take the supremum over all $\Omega' \Subset \Omega$, we obtain

$$
\|\nabla u\|^2_{L^2(\Omega)} \le \frac{2}{\theta}\frac{\|f\|^2_{L^2(\Omega)}}{2} + \big(1 + \|b\|_{L^\infty(\Omega)} + 2\theta\|c\|_{L^\infty(\Omega)}\big)\|u\|^2_{L^2(\Omega)}.
$$

We have thus obtained (207). □

Lemma 16.18. *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ open. Suppose that $u \in H^1(\Omega)$ is such that $Lu = f$ for some $f \in L^2(\Omega)$, where L has $b = 0$ and $c = 0$. Equivalently, $u \in H^1(\Omega)$ and $f \in L^2(\Omega)$ are such that*

$$
(209)\qquad \forall v \in H^1_0(\Omega), \qquad \langle A\nabla u, \nabla v\rangle_{L^2(\Omega)} = \langle f, v\rangle_{L^2(\Omega)},
$$

where $A \in L^\infty(\Omega; \mathbb{C}^{n\times n})$ satisfies the ellipticity condition (190).

For every $\Omega'' \Subset \Omega$ there exist constants C and ϵ depending on Ω'', $\|A\|_{L^\infty(\Omega)}$, $\|\mathrm{D}A\|_{L^\infty(\Omega)}$, and θ, such that, for every h with $0 < |h| < \epsilon$, and for every $j \in \{1, \dots, n\}$,

$$
(210)\qquad \|\Delta^h_j \nabla u\|^2_{L^2(\Omega'')} \le C(\|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(\Omega)}).
$$

Proof. Fix $j \in \{1, \dots, n\}$ and $h \ne 0$. Let $\Omega' \Subset \Omega' \Subset \Omega$ and $\zeta \in C^\infty(\Omega; [0,1])$ be such that

$$
\Omega'' \subset \operatorname{int}\{\zeta = 1\} \subset \operatorname{spt}(\zeta) \Subset \Omega' \Subset \Omega.
$$

Then there exists $\epsilon > 0$ such that, for every h with $0 < |h| < \epsilon$,

$$
v := -\Delta^{-h}_j(\zeta^2 \Delta^h_j u) \in H^1_0(\Omega).
$$

With the aim of applying (209) with this test function, we make the following estimates. We see that

$$
\begin{aligned}
\langle A\nabla u, \nabla v\rangle_{L^2(\Omega)} &= \langle A\nabla u, \nabla(-\Delta^{-h}_j(\zeta^2\Delta^h_j u))\rangle_{L^2(\Omega)} \\
&\overset{(154)}{=} \langle \Delta^h_j(A\nabla u), \nabla(\zeta^2\Delta^h_j u)\rangle_{L^2(\Omega)} \\
&\overset{(153)}{=} \langle (\Delta^h_j A)\nabla u + A^{(h)}\Delta^h_j\nabla u, \Delta^h_j u 2\zeta\nabla\zeta + \zeta^2\nabla(\Delta^h_j u)\rangle_{L^2(\Omega)} \\
&= F_1 + F_2,
\end{aligned}
$$

where $A^{(h)}(x) = A(x + he_j)$ and

$$
\begin{aligned}
F_1 &= \langle A^{(h)}\Delta^h_j\nabla u, \zeta^2\nabla(\Delta^h_j u)\rangle_{L^2(\Omega)} \\
&= \langle A^{(h)}\zeta\Delta^h_j\nabla u, \zeta\Delta^h_j\nabla u\rangle_{L^2(\Omega)} \\
&\overset{(190)}{\ge} \theta\|\zeta\Delta^h_j\nabla u\|_{L^2(\Omega)} \\
&\ge \theta\|\Delta^h_j\nabla u\|_{L^2(\Omega'')}; \\
F_2 &= \langle (\Delta^h_j A)\nabla u, \Delta^h_j u 2\zeta\nabla\zeta\rangle_{L^2(\Omega)} \\
&\quad + \langle (\Delta^h_j A)\nabla u, \zeta^2\nabla(\Delta^h_j u)\rangle_{L^2(\Omega)} \\
&\quad + \langle A^{(h)}\Delta^h_j\nabla u, \Delta^h_j u 2\zeta\nabla\zeta\rangle_{L^2(\Omega)} \\
&\le 2\|\Delta^h_j A\|_{L^\infty(\Omega')}\|\nabla u\|_{L^2(\Omega')}\|\Delta^h_j u\|_{L^2(\Omega')}\|\nabla\zeta\|_{L^\infty(\Omega')}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \|\Delta_j^h A\|_{L^\infty(\Omega')} \|\nabla u\|_{L^2(\Omega')} \|\zeta \Delta_j^h \nabla u\|_{L^2(\Omega')} \\
&\quad + 2\|A\|_{L^\infty(\Omega')} \|\zeta \Delta_j^h \nabla u\|_{L^2(\Omega')} \|\Delta_j^h u\|_{L^2(\Omega')} \|\nabla \zeta\|_{L^\infty(\Omega')} \\
&\le 2\|\mathrm{D}A\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 \|\nabla \zeta\|_{L^\infty(\Omega)} \\
&\quad + \|\mathrm{D}A\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\zeta \Delta_j^h \nabla u\|_{L^2(\Omega)} \\
&\quad + 2\|A\|_{L^\infty(\Omega)} \|\zeta \Delta_j^h \nabla u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla \zeta\|_{L^\infty(\Omega)} \\
&\overset{(197)}{\le} 2\|\mathrm{D}A\|_{L^\infty(\Omega)} \|\nabla \zeta\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 \\
&\quad + 4\|\nabla \zeta\|_{L^\infty(\Omega)}^2 \frac{\|A\|_{L^\infty(\Omega)}^2 + \|\mathrm{D}A\|_{L^\infty(\Omega)}^2}{\theta} \|\nabla u\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\theta}{2} \|\zeta \Delta_j^h \nabla u\|_{L^2(\Omega)}^2.
\end{aligned}
$$

Since

$$
\langle A\nabla u, \nabla v\rangle_{L^2(\Omega)} = \langle u, f\rangle_{L^2(\Omega)} \le \|u\|_{L^2(\Omega)} \|f\|_{L^2(\Omega)} \overset{(197)}{\le} \frac{\|u\|_{L^2(\Omega)}^2}{2} + \frac{\|f\|_{L^2(\Omega)}^2}{2},
$$

we obtain

$$
\begin{aligned}
\frac{\theta}{2} \|\zeta \Delta_j^h \nabla u\|_{L^2(\Omega)}^2 \le{}& \frac{\|u\|_{L^2(\Omega)}^2}{2} + \frac{\|f\|_{L^2(\Omega)}^2}{2} \\
&+ 2\|\mathrm{D}A\|_{L^\infty(\Omega)} \|\nabla \zeta\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)}^2 \\
&+ 4\|\nabla \zeta\|_{L^\infty(\Omega)}^2 \frac{\|A\|_{L^\infty(\Omega)}^2 + \|\mathrm{D}A\|_{L^\infty(\Omega)}^2}{\theta} \|\nabla u\|_{L^2(\Omega)}^2.
\end{aligned}
$$

We conclude that there is a constant $C \in \mathbb{R}$ depending on $\|A\|_{L^\infty(\Omega)}$, $\|\mathrm{D}A\|_{L^\infty(\Omega)}$, $\|\nabla\zeta\|_{L^\infty(\Omega)}$ and θ such that

$$
\|\zeta \Delta_j^h \nabla u\|_{L^2(\Omega)}^2 \le C(\|u\|_{L^2(\Omega)}^2 + \|\nabla u\|_{L^2(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2).
$$

Since $\zeta = 1$ on Ω'', then we get (210). □

Theorem 16.19 (Regularity of solutions to Elliptic PDE)**.** *Let $n \ge 2$ and $\Omega \subset \mathbb{R}^n$ open.*

Let L be a differential operator as in (191)*, with bounded coefficients* (190)*, and satisfying the ellipticity condition* (189)*. Moreover, we assume that*

$$
\|\mathrm{D}A\|_{L^\infty(\Omega)} < \infty,
$$

that is, that A is Lipschitz.

Then, for every $\Omega' \Subset \Omega$ there exists a constant C such that

$$
\forall u \in H^1(\Omega), \qquad \|\mathrm{D}^2 u\|_{L^2(\Omega')} \le C(\|u\|_{L^2(\Omega)} + \|Lu\|_{L^2(\Omega)}). \tag{211}
$$

In particular, if $u \in H^1(\Omega)$ is such that $Lu \in L^2(\Omega)$, then $u \in H^2_{\mathrm{loc}}(\Omega)$.

Proof. Let $\Omega' \Subset \Omega$ and $u \in H^1(\Omega)$ with $f = Lu \in L^2(\Omega)$ (if $\|Lu\|_{L^2(\Omega)} = \infty$, then (214) is trivial. Define M as the differential operator

$$
\forall w \in H^1(\Omega), \qquad Mw = -\operatorname{div}(A\nabla w) = Lw - (b \cdot \nabla w + cw).
$$

Notice that, if $w \in H^1(\Omega)$, then $b \cdot \nabla w + cw \in L^2(\Omega)$. So,

$$
Mu = \tilde f := f - (b \cdot \nabla u + cu) \in L^2(\Omega).
$$

Notice that

$$
\|\tilde f\|_{L^2(\Omega)} \le \|b\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} + \|c\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)}. \tag{212}
$$

Applying first Lemma 16.18, we obtain a constant C (independent of u) such that

$$
\limsup_{h\to 0} \|\Delta_j^h \nabla u\|_{L^2(\Omega'')} \le C(\|u\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)} + \|\tilde f\|_{L^2(\Omega)})
$$

The estimate (212) and the Caccioppoli Inequality of Lemma 16.17, then implies that there exists a constant C (independent of u) such that

$$
\limsup_{h\to 0} \|\Delta_j^h \nabla u\|_{L^2(\Omega'')} \le C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)}).
$$

By Theorem 15.47 (using also the quantitative version given by Proposition 15.43), we obtain (214). □

Exercise 16.20. Re-write Theorem 16.19, together with Lemmata 16.17 and 16.18, and their proofs, for $L = \triangle$ on $\mathbb{R}^n$. ◊

Corollary 16.21 (Regularity of solutions to Elliptic PDE)**.** *Let $n \geq 2$ and $\Omega \subset \mathbb{R}^n$ open.*

Let L be a differential operator as in (191)*, with bounded coefficients* (190)*, and satisfying the ellipticity condition* (189)*. Moreover, we assume that, for some $m \in \mathbb{N}$, we have*

$$A \in W^{m+1,\infty}(\Omega; \mathbb{C}^{n\times n}), \qquad b \in W^{m,\infty}(\Omega; \mathbb{C}^n), \qquad c \in W^{m,\infty}(\Omega; \mathbb{C}). \tag{213}$$

Then, for every $\Omega' \Subset \Omega$ there exists a constant C such that

$$\forall u \in H^1(\Omega), \qquad \|u\|_{H^{m+2}(\Omega')} \leq C(\|u\|_{L^2(\Omega)} + \|Lu\|_{H^m(\Omega)}). \tag{214}$$

In particular, if $u \in H^1(\Omega)$ is such that $Lu \in H^m(\Omega)$, then $u \in H^{m+2}_{\mathrm{loc}}(\Omega)$.

Proof. We prove the statement by induction over m. Let $\ell \in \mathbb{N}$. If $\ell = 0$, then we just apply Theorem 16.19. Suppose $\ell > 0$ and suppose that the statement above holds for $m < \ell$: we will prove it for $m = \ell$. Under the hypothesis above on L, we have that for every $\Omega' \Subset \Omega$ there exists a constant C such that

$$\forall u \in H^1(\Omega), \qquad \|u\|_{H^{\ell+1}(\Omega')} \leq C(\|u\|_{L^2(\Omega)} + \|Lu\|_{H^{\ell-1}(\Omega)}). \tag{215}$$

For each $j \in \{1, \dots, n\}$, we have

$$\begin{aligned} L\partial_j u &= -\operatorname{div}(A\nabla\partial_j u) + b\cdot\nabla\partial_j u + c\partial_j u \\ &= \partial_j(-\operatorname{div}(A\nabla u) + b\cdot\nabla u + cu) - (-\operatorname{div}(\partial_j A\nabla u) + \partial_j b\cdot\nabla u + \partial_j cu) \\ &= \partial_j Lu - L_j u. \end{aligned} \tag{216}$$

Straightforward estimates give, for every $m \in \mathbb{N}$ for which (213) holds, we have $C'_m < \infty$ such that

$$\begin{aligned} \|L_j u\|_{H^m(\Omega)} &\leq \|\operatorname{div}(\partial_j A\nabla u)\|_{H^m(\Omega)} + \|\partial_j b\cdot\nabla u\|_{H^m(\Omega)} + \|\partial_j cu\|_{H^m(\Omega)} \\ &\leq \sum_{|\alpha|\leq m} \|\mathrm{D}^\alpha(\operatorname{div}(\partial_j A\nabla u))\|_{L^2(\Omega)} + \|\mathrm{D}^\alpha\partial_j b\cdot\nabla u\|_{L^2(\Omega)} + \|\mathrm{D}^\alpha\partial_j cu\|_{L^2(\Omega)} \\ &\leq C_m(\|A\|_{W^{m+2,\infty}(\Omega)} + \|b\|_{W^{m+1,\infty}(\Omega)} + \|c\|_{W^{m+1,\infty}(\Omega)})\|u\|_{H^{m+2}(\Omega)} \\ &\overset{(213)}{\leq} C'_m\|u\|_{H^{m+2}(\Omega)}. \end{aligned} \tag{217}$$

So

$$\begin{aligned} \|u\|_{H^{\ell+2}(\Omega')} &\leq \|u\|_{L^2(\Omega)} + \sum_{j=1}^n \|\partial_j u\|_{H^{\ell+1}(\Omega')} \\ &\overset{(215)}{\leq} \|u\|_{L^2(\Omega)} + C\sum_{j=1}^n(\|\partial_j u\|_{L^2(\Omega)} + \|L\partial_j u\|_{H^{\ell-1}(\Omega')}) \\ &\overset{(216)}{\leq} \|u\|_{L^2(\Omega)} + C\sum_{j=1}^n(\|\partial_j u\|_{L^2(\Omega)} + \|\partial_j Lu\|_{H^{\ell-1}(\Omega')} + \|L_j u\|_{H^{\ell-1}(\Omega')}) \\ &\overset{(217)}{\leq} C'\left(\|u\|_{H^1(\Omega)} + \|Lu\|_{H^\ell(\Omega)} + \|u\|_{H^{\ell+1}(\Omega)}\right) \\ &\overset{(215)}{\leq} C''\left(\|Lu\|_{H^\ell(\Omega)} + \|u\|_{L^2(\Omega)} + \|Lu\|_{H^{\ell-1}(\Omega)}\right) \\ &\leq C'''\left(\|Lu\|_{H^\ell(\Omega)} + \|u\|_{L^2(\Omega)}\right). \end{aligned}$$

We eventually have (214) for $m = \ell$. □

17. Exercises

§17.1. Eigenvalues of laplacian on interval. We want to study, for $\lambda \in \mathbb{C}$ and $\ell > 0$,

$$\begin{cases} -\triangle u + \lambda u = 0 & \text{in } (0,\ell), \\ u(0) = u(\ell) = 0. \end{cases}$$

In other words,

$$\begin{cases} u'' = \lambda u & \text{in } (0,\ell), \\ u \in H^1((0,\ell)). \end{cases} \tag{218}$$

Lemma 17.1. *Let u be a solution to* (218). *Then $u \in C^\infty((0,\ell))$.*

Proof. This is an application of Corollary 16.21. □

Be aware that the regularity stated in Lemma 17.1 is in the open interval $(0,\ell)$ and not on the whole $[0,\ell]$. We will procede as follows:

(1) the assumption $u \in H^1_0((0,\ell))$ alone gives us that u is continuous on $[0,\ell]$ and that $u(0) = u(\ell)$; see the following two Lemmata 17.3 and 17.4.
(2) Then, we will extend u to $\mathbb{R}$ in such a way that the extension $\tilde{u} = E(u)$ is a bounded function on $\mathbb{R}$ that solves $\triangle \tilde{u} = \lambda \tilde{u}$ on $\mathbb{R}$; see Lemma 17.5 and Corollary 17.6.
(3) We will find all solutions of $\triangle v = \lambda v$ for $v \in \mathscr{S}'(\mathbb{R})$ using the Fourier transform; see Proposition 17.7.
(4) Among the solutions v on $\mathbb{R}$, we look for those v that are extensions of functions in $H^1_0((0,\ell))$; see Proposition 17.9. In this way, we will find all solutions to (218).

Exercise 17.2. Try to guess solutions to (218). Try also to show they are the only one. ♢

§17.1.1. *Fundamental properties of u.*

Lemma 17.3. *If $u \in W^{1,1}((0,\ell))$, then, up to changing u on a set of measure zero, $u \in C([0,\ell])$ and*

$$\forall x \in [0,\ell], \qquad u(x) = u(0) + \int_0^x u'(y)\,\mathrm{d}y.$$

Proof. Define $v : (0,\ell) \to \mathbb{C}$ by

$$v(x) = \int_{\ell/2}^x u'(y)\,\mathrm{d}y.$$

We claim that

$$v' = u'. \tag{219}$$

If $\phi \in C^\infty_c((0,\ell))$, then

$$\begin{aligned} \int_0^\ell v(x)\phi'(x)\,\mathrm{d}x &= \int_0^\ell \int_{\ell/2}^x u'(y)\phi'(x)\,\mathrm{d}y\,\mathrm{d}x \\ &= -\int_0^{\ell/2}\int_0^y u'(y)\phi'(x)\,\mathrm{d}x\,\mathrm{d}y + \int_{\ell/2}^\ell \int_y^\ell u'(y)\phi'(x)\,\mathrm{d}x\,\mathrm{d}y \\ &= -\int_0^\ell u'(y)\phi(y)\,\mathrm{d}y \\ &= \int_0^\ell u(y)\phi'(y)\,\mathrm{d}y. \end{aligned}$$

This shows (219).

Therefore, $(u-v)' = 0$ and thus, by the Constancy Theorem 13.62, there exists $c \in \mathbb{C}$ such that, for almost every $x \in (0,\ell)$, $u(x) = c + v(x)$. We conclude that, up to changing u on a set of measure zero,

$$u(x) = u(\ell/2) + \int_{\ell/2}^x u'(y)\,\mathrm{d}y.$$

By the continuity of the integral, not only u is continuous, but also $\lim_{x\to 0} u(x)$ exists and it is equal to $u(\ell/2) - \int_0^{\ell/2} u'(y)\,\mathrm{d}y$, and similarly for $\lim_{x\to \ell} u(x)$. □

Lemma 17.4. *Let $v \in H^1((0,\ell)) = W^{1,2}((0,\ell))$. Then $v \in W^{1,1}((0,\ell)) \cap C([0,\ell])$. Moreover, if $v \in H_0^1((0,\ell))$, then $v(0) = v(\ell) = 0$.*

Proof. Since $L^2((0,\ell)) \subset L^1((0,\ell))$, then $v \in W^{1,1}((0,\ell))$. Lemma 17.3 implies that, up to changing v on a set of measure zero, $v \in C([0,\ell])$.

Suppose $v \in H_0^1((0,\ell))$. By definition of $H_0^1((0,\ell))$, there exists a sequence $\{v_j\}_{j\in\mathbb{N}} \subset C_c^\infty((0,\ell))$ such that $v_j \to v$ in $H^1((0,\ell))$, that is, $\lim_{j\to\infty} \|v - v_j\|_{L^2((0,\ell))} = 0$ and $\lim_{j\to\infty} \|v' - v_j'\|_{L^2((0,\ell))} = 0$. Since the Hölder inequality implies $\|f\|_{L^1((0,\ell))} \le \sqrt{\ell}\|f\|_{L^2((0,\ell))}$, we obtain $\lim_{j\to\infty} \|v - v_j\|_{L^1((0,\ell))} = 0$ and $\lim_{j\to\infty} \|v' - v_j'\|_{L^1((0,\ell))} = 0$. We know then that, up to passing to a subsequence, there exists a set $I \subset (0,\ell)$ of full measure so that $\lim_{j\to\infty} v_j(x) = v(x)$ for every $x \in I$.

We claim that in fact the convergence $v_j \to v$ is pointwise on $[0,\ell]$ (in fact, we prove that it is uniform). Indeed, if $\bar{x} \in I$ and $x \in [0,\ell]$, then, for every $j \in \mathbb{N}$,

$$\begin{aligned} |v(x) - v_j(x)| &= \left| v(\bar{x}) - v_j(\bar{x}) + \int_{\bar{x}}^x (v'(y) - v_j'(y))\,\mathrm{d}y \right| \\ &\le |v(\bar{x}) - v_j(\bar{x})| + \int_0^\ell |v'(y) - v_j'(y)|\,\mathrm{d}y. \end{aligned}$$

Therefore, $\lim_{j\to\infty} |v(x) - v_j(x)| = 0$ for every $x \in [0,\ell]$.

Since $v_j(0) = v_j(\ell) = 0$ for all j, then $v(0) = v(\ell) = 0$. □

§17.1.2. *Extension of u to $\mathbb{R}$.*

Lemma 17.5. *For $v \in L^1((0,\ell))$, define $E(v) \in L^1_{\mathrm{loc}}(\mathbb{R})$ as*

$$E(v)(x) = \sum_{k\in\mathbb{Z}} \left(u(x \mod \ell\mathbb{Z}) \mathbb{1}_{\ell[2k,2k+1]}(x) - u(-x \mod \ell\mathbb{Z}) \mathbb{1}_{\ell[2k+1,2k+2]}(x) \right).$$

Then, distributionally, for every $v \in W^{1,1}((0,\ell))$,

$$E(v') = E(v)'.$$

Proof. From Lemma 17.3, we know that, up to modifying v on a set of measure zero, $v \in C([0,\ell])$ and, for all $x \in [0,\ell]$,

$$v(x) = v(0) + \int_0^x v'(y)\,\mathrm{d}y. \tag{220}$$

The identity (220) implies that, for every $\phi \in \mathscr{D}(\mathbb{R})$,

$$\int_0^\ell u(x)\phi'(x)\,\mathrm{d}x = u(\ell)\phi(\ell) - u(0)\phi(0) - \int_0^\ell u'(x)\phi(x)\,\mathrm{d}x. \tag{221}$$

Let $\phi \in \mathscr{D}(\mathbb{R})$.

$$\begin{aligned} -\int_{\mathbb{R}} E(v)'(x)\phi(x)\,\mathrm{d}x &\overset{\mathrm{def}}{=} \int_{\mathbb{R}} E(v)(x)\phi'(x)\,\mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \int_0^\ell \left(u(x)\phi'(2k\ell + x) - u(\ell - x)\phi'((2k+1)\ell + x) \right)\,\mathrm{d}x \\ &\overset{(221)}{=} \sum_{k\in\mathbb{Z}} u(\ell)\phi'(2k\ell + \ell) - u(0)\phi'(2k\ell) \\ &\qquad - u(0)\phi'((2k+1)\ell + \ell) + u(\ell)\phi'((2k+1)\ell) \\ &\qquad - \int_0^\ell \left(u'(x)\phi(2k\ell + x) - u'(\ell - x)\phi((2k+1)\ell + x) \right)\,\mathrm{d}x \\ &\overset{\mathrm{def}}{=} -\int_{\mathbb{R}} E(v')(x)\phi(x)\,\mathrm{d}x. \end{aligned}$$

□

Corollary 17.6. *Let u be a solution to (218). Let $\tilde{u} = E(u)$ be the extension of u as in Lemma 17.5. Then*

$$\triangle \tilde{u} = \lambda \tilde{u} \text{ in } \mathbb{R}. \tag{222}$$

Proof. If $u \in H^1_0((0,\ell))$, then $u, u' \in W^{1,1}((0,\ell)) \cap C([0,\ell])$ by Lemma **??**. Therefore, by Lemma 17.5, we have the following identities of distributions on $\mathbb{R}$:

$$\lambda E(u) = E(\lambda u) = E(u'') = E(u')' = E(u)''.$$

This is (222). □

§17.1.3. *Solve $v'' = \lambda v$ in $\mathbb{R}$.*

Proposition 17.7. *Let $\lambda \in \mathbb{C}$. A Schwartz distribution $v \in \mathscr{S}'(\mathbb{R}) \setminus \{0\}$ solves*

$$v'' = \lambda v \tag{223}$$

if and only if $\lambda = 0$ and v is affine, or $\lambda \in (-\infty, 0)$ and there are $a, b \in \mathbb{C}$ with

$$v = a\exp(i\sqrt{-\lambda}x) + b\exp(-i\sqrt{-\lambda}x) = a\exp(i\mu x) + b\exp(-i\mu x), \tag{224}$$

wher $\mu > 0$ and $\lambda = -\mu^2$.

Proof. (224) ⇒ (223). This just a direct computation, see Exercise 17.8

(223) ⇒ (224). Apply the Fourier transform to both sides of (223), to obtain

$$(2\pi i\xi)^2 \mathscr{F}(v) \overset{(135)}{=} \mathscr{F}(v'') \overset{(223)}{=} \mathscr{F}(\lambda v) = \lambda \mathscr{F}(v).$$

This means that, for every $\phi \in \mathscr{S}(\mathbb{R})$,

$$0 = {}_{\mathscr{S}'}\langle (2\pi i\xi)^2 \mathscr{F}(v) - \lambda \mathscr{F}(v) | \phi \rangle_{\mathscr{S}} = {}_{\mathscr{S}'}\langle \mathscr{F}(v) | (-4\pi^2\xi^2 - \lambda)\phi \rangle_{\mathscr{S}}. \tag{225}$$

It follows that $\mathscr{F}(v)$ is supported on $\{\xi \in \mathbb{R} : 4\pi^2\xi^2 + \lambda = 0\}$. So, if $\lambda \notin (-\infty, 0]$, then $\operatorname{spt}(\mathscr{F}(v)) = \emptyset$ and thus $v = 0$. If $\lambda = 0$, then $v'' = 0$ and we know that v must be affine; see Exercise 10.1.

Suppose $\lambda \in (-\infty, 0)$, and let $\mu > 0$ such that

$$\lambda = -\mu^2.$$

Thus, we have $\sqrt{-\frac{\lambda}{4\pi^2}} = \frac{\mu}{2\pi}$ The above statement about the support of $\mathscr{F}(v)$ now reads

$$\operatorname{spt}(\mathscr{F}(v)) \subset \left\{ \frac{\mu}{2\pi}, -\frac{\mu}{2\pi} \right\}.$$

By Proposition 13.37,

$$\mathscr{F}(v) = \sum_{\alpha=0}^{\infty} \left(a_\alpha \partial^\alpha \delta_{\frac{\mu}{2\pi}} + b_\alpha \partial^\alpha \delta_{-\frac{\mu}{2\pi}} \right).$$

We use again (225) to obtain that, for every $\phi \in \mathscr{S}'(\mathbb{R})$,

$$\begin{aligned}
0 &= {}_{\mathscr{S}'}\langle \mathscr{F}(v) | (-4\pi^2\xi^2 - \lambda)\phi \rangle_{\mathscr{S}} \\
&= \sum_{\alpha=0}^{\infty} \left(a_\alpha \partial^\alpha \delta_{\frac{\mu}{2\pi}} [(-4\pi^2\xi^2 - \lambda)\phi] + b_\alpha \partial^\alpha \delta_{-\frac{\mu}{2\pi}} [(-4\pi^2\xi^2 - \lambda)\phi] \right) \\
&= \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha} \binom{\alpha}{\beta} \left(a_\alpha \delta_{\frac{\mu}{2\pi}} [\partial^\beta (4\pi^2\xi^2 + \lambda) \partial^{\alpha-\beta}\phi] + b_\alpha \delta_{-\frac{\mu}{2\pi}} [\partial^\beta (4\pi^2\xi^2 + \lambda) \partial^{\alpha-\beta}\phi] \right) \\
&\overset{(*)}{=} \sum_{\alpha=1}^{\infty} \sum_{\beta=1}^{\alpha} \binom{\alpha}{\beta} \left(a_\alpha \delta_{\frac{\mu}{2\pi}} [\partial^\beta (4\pi^2\xi^2 + \lambda) \partial^{\alpha-\beta}\phi] + b_\alpha \delta_{-\frac{\mu}{2\pi}} [\partial^\beta (4\pi^2\xi^2 + \lambda) \partial^{\alpha-\beta}\phi] \right) \\
&= a_1 8\pi^2 \frac{\mu}{2\pi} \phi(\frac{\mu}{2\pi}) - b_1 8\pi^2 \frac{\mu}{2\pi} \phi(-\frac{\mu}{2\pi}) \\
&\quad + \sum_{\alpha=2}^{\infty} \Bigg(a_\alpha (\alpha 8\pi^2 \frac{\mu}{2\pi} \partial^{\alpha-1}\phi(\frac{\mu}{2\pi}) + \alpha(\alpha-1) 8\pi^2 \partial^{\alpha-2}\phi(\frac{\mu}{2\pi})) \\
&\qquad + b_\alpha (-\alpha 8\pi^2 \frac{\mu}{2\pi} \partial^{\alpha-1}\phi(-\frac{\mu}{2\pi}) + \alpha(\alpha-1) 8\pi^2 \partial^{\alpha-2}\phi(-\frac{\mu}{2\pi})) \Bigg),
\end{aligned}$$

where in $(*)$ we observed that, if $\beta = 0$, then the summand is zero. Since ϕ is an arbitrary function in $\mathscr{S}(\mathbb{R})$, we obtain $a_\alpha = b_\alpha = 0$ for all $\alpha \geq 1$. Hence,

$$\mathscr{F}(v) = a_0 \delta_{\frac{\mu}{2\pi}} + b_0 \delta_{-\frac{\mu}{2\pi}}.$$

From Exercise 14.26, we obtain

$$\begin{aligned} v(x) &\overset{(136)}{=} a_0 \exp(2\pi i \frac{\mu}{2\pi} x) + b_0 \exp(-2\pi i \frac{\mu}{2\pi} x) \\ &= a_0 \exp(i\mu x) + b_0 \exp(-i\mu x). \end{aligned}$$

□

Exercise 17.8. Show (224) $\Rightarrow$ (223) in Proposition 17.7. ◊

Proposition 17.7 states that the spectrum of $-\partial^2 = -\triangle$ on $\mathbb{R}$ is $[0, +\infty)$:

$$\sigma(-\triangle) = [0, +\infty).$$

§17.1.4. *Select solutions that are extensions.*

Proposition 17.9. *A non-zero function* $u \in H_0^1((0,\ell))$ *solves* (218) *if and only if the following two hold*

(1) there is $k \in \mathbb{N} \setminus \{0\}$ *such that* $\lambda = -\left(\frac{\pi}{\ell} k\right)^2$, *and*
(2) there is $a \in \mathbb{C} \setminus \{0\}$ *such that*

$$u(x) = a \left(\exp\left(i \frac{k\pi}{\ell} x \right) - \exp\left(-i \frac{k\pi}{\ell} x \right) \right) = 2ia \sin\left(\frac{k\pi}{\ell} x \right).$$

Proof. If u solves (218), then its extension $\tilde{u} = E(u)$ defined in Lemma 17.5 is of the form given by Proposition 17.7, with the additional boundary conditions $\tilde{u}(0) = \tilde{u}(\ell) = 0$. If $\tilde{u}$ is affine, then $\tilde{u} = 0$. Therefore, if $\tilde{u} \neq 0$, then there are $a, b \in \mathbb{C}$ with (224), i.e.,

$$\tilde{u}(x) = a \exp(i\mu x) + b \exp(-i\mu x),$$

where $\mu > 0$ and $\lambda = -\mu^2$.

The condition $\tilde{u}(0) = 0$ implies $b = -a$. The condition $\tilde{u}(\ell) = 0$, implies

$$0 = a \exp(i\mu\ell) - a \exp(-i\mu\ell) = a \exp(i\mu\ell)(1 - \exp(-2i\mu\ell)).$$

Therefore, either $a = v = 0$, or $2\mu\ell \in 2\pi\mathbb{Z}$, i.e., $\mu\ell \in \pi\mathbb{Z}$, i.e., $\sqrt{-\lambda} = \mu \in \frac{\pi}{\ell}\mathbb{N}$. □

§17.2. **More about the extension operator** E**.** Lemma 17.5 gives a linear operator $E : L^1((0,\ell)) \to L^1_{\text{loc}}(\mathbb{R})$.

Part 5. Extras

18. Comments

§18.1. Correct proof of Theorem 10.32.

Correct proof of Theorem 10.32. Let $U \subset \mathbb{R}^n$ be open and $u \in C^\infty(U)$ a harmonic function. Fix $\hat{x} \in U$ and set $\hat{r} = \frac{1}{4}\mathrm{dist}(\hat{x}, \partial U)$. We claim that there exists $\epsilon \in (0,1)$ such that, if

$$r < \epsilon \hat{r},$$

then the Taylor series of u centered at $\hat{x}$ converges on $B(\hat{x}, r)$ to u, that is, for every $x \in B(\hat{x}, r)$,

$$u(x) = \lim_{N\to\infty} \sum_{k=0}^{N} \sum_{|\alpha|=k} \frac{\mathrm{D}^\alpha u(\hat{x})}{\alpha!}(x-\hat{x})^\alpha .$$

To this aim, define the reminder function

$$R_N(x) = u(x) - \sum_{k=0}^{N-1} \sum_{|\alpha|=k} \frac{\mathrm{D}^\alpha u(\hat{x})}{\alpha!}(x-\hat{x})^\alpha .$$

For every $x \in B(\hat{x}, r)$ there exists $t_x \in [0,1]$ such that

$$R_N(x) = \sum_{|\alpha|=N} \frac{\mathrm{D}^\alpha u(\hat{x} + t(x-\hat{x}))}{\alpha!}(x-\hat{x})^\alpha .$$

Using Proposition 10.27, we make the following estimate: since $\hat{x} + t(x-\hat{x}) \in B(\hat{x}, r) \subset B(\hat{x}, \hat{r})$, then $B(\hat{x} + t(x-\hat{x}), \hat{r}) \subset B(\hat{x}, 2\hat{r}) \subset U$. Therefore,

$$\begin{aligned}
|R_N(x)| &\le \sum_{|\alpha|=N} \frac{|\mathrm{D}^\alpha u(\hat{x} + t(x-\hat{x}))|}{\alpha!}|x-\hat{x}|^N \\
&\overset{(42)}{\le} \frac{(2^{n+1}nN)^N \|u\|_{L^1(B(\hat{x}+t(x-\hat{x}),\hat{r}))}}{\omega_n \hat{r}^{n+N}} \sum_{|\alpha|=N} \frac{1}{\alpha!}|x-\hat{x}|^N \\
&\le \frac{(2^{n+1}nN)^N \|u\|_{L^1(B(\hat{x},2\hat{r}))}}{\omega_n \hat{r}^{n+N}} r^N \sum_{|\alpha|=N} \frac{1}{\alpha!} \\
&\overset{(226)}{=} \frac{\|u\|_{L^1(B(\hat{x},\hat{r}))}}{\omega_n \hat{r}^n} \epsilon^N (2^{n+1}nN)^N \frac{n^N}{N!} \\
&= \frac{\|u\|_{L^1(B(\hat{x},\hat{r}))}}{\omega_n \hat{r}^n} \frac{\sqrt{2\pi N}(N/e)^N}{N!} \frac{(\epsilon 2^{n+1} n^2 N)^N}{\sqrt{2\pi N}(N/e)^N} \\
&= \frac{\|u\|_{L^1(B(\hat{x},\hat{r}))}}{\omega_n \hat{r}^n} \frac{\sqrt{2\pi N}(N/e)^N}{N!} \frac{(\epsilon 2^{n+1} n^2 e)^N}{\sqrt{2\pi N}},
\end{aligned}$$

where we have used the Multinomial Theorem

$$n^N = (\sum_{j=1}^{n} 1)^N = \sum_{|\alpha|=N} \frac{N!}{\alpha!}. \tag{226}$$

Using Stirling's formula

$$\lim_{N\to\infty} \frac{\sqrt{2\pi N}(N/e)^N}{N!} = 1,$$

we conclude that, if $\epsilon \in (0,1)$ is so that $\epsilon 2^{n+1} n^2 e < 1$, then $\lim_{N\to\infty} |R_N(x)| = 0$. □

19. List of Notations

$C^k(\Omega)$	: Functions $\Omega \to \mathbb{C}$ that are smooth up to order k
$C_c^k(\Omega)$	: Functions in $C^k(\Omega)$ that have compact support contained in Ω
$C_b^k(\Omega)$	: Bounded functions belonging to $C^k(\Omega)$
$C^k(\Omega; W)$, $C_c^k(\Omega; W)$	: Functions $\Omega \to W$ with the required regularity
$\langle a, b \rangle$	: Hermitian product: see Section **§1.2**
$a \cdot b$	: Dot product: see Section **§1.1**
$\langle a \vert b \rangle$	: Pairing: see Section **§1.3**
${}_{V^*}\langle a \vert b \rangle_V$	: Pairing: see Section **§1.3**
$\mathrm{D}f$, $\mathrm{D}^\alpha f$, $\mathrm{D}_x^\alpha f$	: Derivatives: see Section 2
∇f	: Gradient: see Section 2
$\partial_x f$, $\partial_x^\alpha f$	: Derivatives: see Section 2
$\Vert u \Vert_{L^p}$, $\Vert u \Vert_{L^p(X)}$, $\Vert u \Vert_{L^p(\mu)}$	: L^p spaces: see Section **§3.1**
$\triangle u$	: Laplace operator: see Section **§10.1**
$\vert\alpha\vert$ for $\alpha \in \mathbb{N}^n$	: size of multi-index: see Section 2
$\mathrm{spt}(\rho)$	: support of ρ, that is, the closure of $\{x : \rho(x) \neq 0\}$
$B(x, r)$	: open ball with center x and radius r, that is, $\{y : \vert y - x\vert < r\}$
$\bar{B}(x, r)$	: closed ball with center x and radius r, that is, $\{y : \vert y - x\vert \leq r\}$; in metric spaces, $\bar{B}(x, r)$ and $\overline{B(x, r)}$ may be different
$\Box u$	: wave operator $\Box = \partial_t^2 - \triangle$; see Section **§12.1**

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References

[1] R. A. Adams and J. J. F. Fournier. *Sobolev spaces.* English. 2nd ed. Vol. 140. Pure Appl. Math., Academic Press. New York, NY: Academic Press, 2003.

[2] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.

[3] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. Springer, New York, 2011, pp. xiv+599.

[4] H. Brézis and F. Browder. "Partial differential equations in the 20th century". English. In: *Adv. Math.* 135.1 (1998), pp. 76–144.

[5] L. C. Evans. *Partial differential equations.* Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749.

[6] H. Federer. *Geometric measure theory.* Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969, pp. xiv+676.

[7] G. B. Folland. *Introduction to partial differential equations.* Second. Princeton University Press, Princeton, NJ, 1995, pp. xii+324.

[8] G. B. Folland. *Real analysis.* Pure and Applied Mathematics (New York). Modern techniques and their applications, A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1984, pp. xiv+350.

[9] L. Hörmander. *The analysis of linear partial differential operators. I.* Vol. 256. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Distribution theory and Fourier analysis. Springer-Verlag, Berlin, 1983, pp. ix+391.

[10] F. John. *Partial differential equations. 4th ed.* English. Vol. 1. Appl. Math. Sci. Springer, Cham, 1982.

[11] W. Rudin. *Functional analysis.* Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424.

[12] L. Schwartz. *A mathematician grappling with his century. Transl. from the French by Leila Schneps.* English. Basel: Birkhäuser, 2001.

[13] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970, pp. xiv+290.

[14] F. Treves. *Basic linear partial differential equations.* Pure and Applied Mathematics, Vol. 62. Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, pp. xvii+470.

[15] D. Werner. *Funktionalanalysis.* extended. Springer-Verlag, Berlin, 2000, pp. xii+501.